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OPTIMAL FIRE-SUPPORT STRATEGIES

James G. Taylor

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Deterministic optimal control theory is used to study the structure of optimal fire-support policies for some time-sequential tactical allocation problems with combat described by Lanchester-type equations of warfare. Numerous specific problems for determining optimal time-sequential fire-distribution policies for supporting weapon systems are studied. A sequence of one-sided time-sequential tactical allocation problems is examined to study how the optimal fire-support policy depends on the nature of the combat

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model (in particular, maneuver element interactions) and on the quantification of combatant objectives. The modelling of the suppressive effects of supporting weapons and their inclusion in such allocation optimization problems are discussed. The modification of such optimal time-sequential fire-support allocation policies by the inclusion of unit "breakpoints" in the model of combat dynamics is briefly examined. Past work is reviewed and future work suggested.

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral Isham Linder
Superintendent

Jack R. Borsting
Provost

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Prepared by:

JAMES G. TAYLOR, Associate Professor
Department of Operations Research

Reviewed by:

MICHAEL G. SOVEREIGN, Chairman
Department of Operations Research

Released by:

ROBERT FOSSUM
Dean of Research

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1. Introduction.

This report documents the author's research on methodology for the quantitative justification of time-sequential fire-support allocation procedures. Carl von Clausewitz (1780-1831), the influential 19th century German military philosopher, said (see p. 191 of [8]) that if theory caused a more critical study of war, then it had achieved its purpose. Today in the 20th century this is particularly true within the context of military operations research for defense planning. General G. I. Pokrovsky (U.S.S.R.) has similarly stressed (see pp. 12-13 of [27]) the importance of scientific investigation of military principles. We will accordingly investigate the principles of fire-support allocation by the consideration of some idealized problems (see [40], [48]). Although these problems are probably too simple to be taken literally, such analytic investigations of the optimization of fire-support allocation may be used to (1) guide higher resolution studies, (2) identify cause-effect relationships between the structure of optimal allocation policies and modelling assumptions, and (3) test the capabilities of proposed computational methods for time-sequential fire-support allocation optimization problems (for example, Lagrange dynamic programming [29] for discrete-time versions of such problems (see also [22], [26], [47])).

The determination of optimal target allocation strategies for supporting weapon systems† is a major problem of contemporary military operations research. This problem frequently arises, for example, in defense planning studies such as the evaluation of proposed fire-support systems or fire-support mixes (see [24])††. The problem is also of interest to the military tactician so that he

†See [48] for a brief discussion between a "primary" weapon system (or infantry) and a "supporting" weapon system.

††See Appendix A for a further discussion.

may have a clearer understanding of the circumstances under which enemy infantry should be engaged by a supporting weapon system (such as artillery) and those under which "counter-battery" fire is to be preferred. Such tactical allocation problems are of particular relevance in light of the Navy mission of fire support (both by ship gunfire and by carrier-based air). Another important related question for defense planners is, "What are appropriate missions over the course of a campaign for tactical air power?" The answer to this question has far-reaching implications for Navy air forces (both carrier-based and land-based) (and, of course, the Air Force). Recently, the USAF Studies and Analysis Group has been using quantitative methodology [47] in trying to answer such questions.

There is interest at present in the Navy and USMC on various aspects of fire-support allocation and evaluation. Additionally, the problem of optimal time-sequential fire-support strategies is related to campaign analysis and optimal campaign strategies. Currently a research project on campaign analysis (sponsored by OP-96) is underway at NPS, and we have given this work consideration in performing the research at hand.

In the research reported here on the principles of optimal time-sequential fire-support allocation we have built upon the previous research of the investigator [34]-[38] who has studied optimal time-sequential tactics for (1) distribution of fire over enemy target types, (2) selection of target type at which to fire, and (3) regulation of firing rate. By considering several combat scenarios, insights have been gained into such important questions as:

(1) How should supporting fires by distributed over enemy targets?

(2) How should targets be selected?

(3) Do target priorities change over time?

(4) How do force levels affect the optimal time-sequential fire-support policy?

(5) How do the number of target types and the nature of combat attrition processes affect the optimal time-sequential fire-support policy?

(6) How does the nature of the planning horizon (i.e., battle termination conditions) affect the optimal time-sequential fire-support policy?

(7) What is the optimal fire-support mix and how is this affected by tactics?

(8) What are the effects of logistics constraints on such policies?

(9) How do the uncertainty and confusion of combat affect optimal time-sequential fire-support allocation strategies?

(10) How do command and control capabilities affect the optimal time-sequential fire-support policy?

In trying to answer the above questions we have given consideration to the following factors:

(1) combatant objectives (form of criterion functional and valuation of surviving forces),

(2) dynamics of the combat attrition process,

(3) weapon system performance characteristics,

(4) termination conditions of conflict,

(5) force strengths and composition,

(6) type of attrition process,

(7) effects of resource constraints,

(8) range capabilities of weapon systems.

Thus, the determination of optimal target allocation strategies for supporting weapon systems is a major problem of contemporary military operations research. Accordingly, the objectives of this research are to determine optimal fire-support strategies in a time-sequential fashion over the course of combat for several situations of tactical interest and to study the dependence of these strategies on the nature of the combat model. In this work consideration is given to the dynamics of combat.† We emphasize the development of explicit expressions for the optimal fire-support policies here. Such results are important not only for their own sake†† but also for testing the capabilities of proposed computational methods (for example, Lagrange dynamic programming [29] for discrete-time versions of such problems or the Lulejian successive-approximation methodology [23], [26]).

In the research reported here we have continued our study of the effects of modelling assumptions on the structure of optimal time-sequential allocation policies (see [34]-[44]). It is the investigator's opinion (based on his knowledge of both the open literature and participation in the symposium on the "State-of-the-Art of Mathematics in Combat Models" (held at General Research Corporation 14-15 June 1973)) that this topic is at present imperfectly understood by analysts. We feel that an important aspect of our research has been its continuity of effort: the author has been considering quantitative methods for optimizing tactical decisions for five years now, and this has provided valuable perspective for the current research on optimal time-sequential fire-support strategies. The author has also profitted from numerous discussions with military officers (both students at NPS and military analysts) on the topic of optimizing tactical decisions.

2. Research Objectives

The general objective of this research is to develop combat attrition models and optimization techniques to extend the state-of-the-art for the

†This should be contrasted with essentially all the work reviewed in [24] in which no consideration is given to the evolution of the course of battle.

††One can clearly see the dependence of the structure of the optimal time-sequential allocation policy on model form and model parameters.

determination of optimal time-sequential fire-support strategies in various tactical scenarios. The specific objectives of the initial phase of this research are: (1) to determine the optimal time-sequential fire-support policies in several situations of tactical interest, (2) to study the dependence of these policies on the functional form of the model for combat dynamics, (3) to determine the sensitivity of these policies to the functional form of the criterion functional, and (4) to develop methodology for the determination of optimal time-sequential fire-support policies when there are different combat dynamics in different "phases" of a battle.

3. Review of Previous Work

A rather comprehensive review of combat modelling theories (in particular, Lanchester-type models of warfare) and related optimization theories for the examination of time-sequential tactical allocation problems is to be found in the investigator's 1972 NPS technical report (see pp. 21-32 of [35] (see also [38])). In this section we will give a brief overview of past work on optimizing fire-support allocations. More detailed reviews (as related to the subject matter of the appendix in question) are to be found in the appendices of this report.

The determination of optimal time-sequential fire-distribution strategies for supporting weapon systems is a major problem of contemporary military operations research. Early work was done on this problem at RAND in the late 1940's and early 1950's (see [12]) and elsewhere (see [1]). Today the problem of optimal air-war strategies is being extensively studied by a number of organizations (see, for example, [7], [14], [22], [30], [47]). This problem was extensively discussed in the workshop on optimization techniques and combat applications at the 1973 Conference on the State-of-the-Art of Mathematics in Combat Models (see [28]) at which the principal investigator was an invited speaker.

Rufus Isaacs considered Arnold Mengel's "War of Attrition and Attack" (see [12]) in Isaacs' now classic book on differential games [18]. Discrete-time versions of this problem of the determination of optimal "air-war" strategies (see also [2], [36], [37]) have been considered by a number of workers as time-sequential combat games [3], [4], [10] (see also [5], [9]). Another related problem was considered by Weiss [48], who studied the optimal selection of targets for engagement by a supporting weapon system.† More recently, Kawara [20] has studied optimal time-sequential fire-distribution strategies for supporting weapon systems in an attack scenario which is a variation of the model considered by Weiss [48]. Other recent work has considered various conceptual and computational aspects of time-sequential combat games [29], [30], [31]. References to the numerous contributions in this field of the principal investigator are to be found in [38] (see also [42], [43]).

4. Research Approach

Our research approach has been to combine Lanchester-type models of warfare with generalized control theory (i.e., optimization theory for dynamic systems (see [16], [17])). This research program has been described in more detail elsewhere [35], [36]. In the initial phase of research reported here we have examined a sequence of one-sided†† time-sequential allocation problems in order to study the dependence of the optimal fire-support policy on the nature of the combat model and on the quantification of combatant objectives.

†See [44], however, for a justification of the optimality of strategies determined by Weiss [48]. A general solution algorithm is also presented in this paper [44].

††In other words, only one of the combatants is free to choose his time-sequential fire-support allocation policy.

Many of these problems represent various versions of the same basic fire-support situation, and many of these optimal control problems have been solved in detail.† The structure of optimal time-sequential fire-support allocation strategies has been studied by considering the solutions for specific optimization problems and comparing and contrasting these. In this work we have used existing methodologies for the modelling of supporting weapon systems (see [32], [45] (also pp. 141-162 of [6])).

In future work, we would extend these results to two-sided optimization problems (i.e., time-sequential combat games).†† Additionally, the effects of the information structure (e.g., whether or not enemy force levels are modelled as being known with certainty) and of modelling "breakpoints" (see [15], [33], [38], [49]) on optimal time-sequential fire-support strategies should be examined in the future.

5. Guided Tour of the Appendices

The organization of this report is to discuss results in general terms in the main body and to leave supporting details for the appendices. Accordingly, we summarize in this section the work which is contained in the appendices and explain why this work was done. The results reported here may be considered to be extensions of our previous work on optimal time-sequential fire-distribution strategies [34]-[38]. Moreover, the work at hand lays the foundation for more extensive work on the quantitative analysis of time-sequential fire-support allocation and on applications of generalized control theory to problems of military operations research.

†Such results are useful for evaluating computational algorithms (see above).

††R. Isaacs [19] has emphasized, however, the difficulties attendant with the transition from one-sided to two-sided dynamic optimization problems.

In Appendix A we consider a sequence of simplified models in order to study the effect of the nature of dynamic combat interactions on optimal time-sequential fire-support allocation policies and gain insights into their structure. First we consider a general one-sided,[†] time-sequential fire-support allocation problem, and then we consider various particularizations (in all, ten) of this general problem. These time-sequential fire-support allocation problems are solved by applying the mathematical theory of optimal control. In this work we emphasize developing "closed-form" solutions in order to be able to conveniently see the structure of optimal fire-support policies without spending the time and effort of extensive numerical determinations. Moreover, part of our research has been to determine idealized problems that are still militarily realistic but yet amenable to (at least partially) "closed-form" solution. By contrasting the structures of the optimal time-sequential fire-support policies for these various problems we study the dependence of these policies on the functional form of the model of combat dynamics. Additionally, we consider the effect of suppression (see, for example, [21]) on such optimal fire-support policies. We review different ways in which to model suppressive effects within the context of Lanchester-type formulations and briefly consider two fire-support allocation problems with suppressive effects included in the model of combat dynamics.

The research reported in Appendix A was undertaken to develop an understanding of the dependence of optimal fire-support allocation policies on the nature of the combat model (i.e., the mathematical form of the model for combat attrition). We were interested in trying to provide insights into

[†]Only one of the two combatants is free to choose his time-sequential fire-support policy.

the answers to the following questions:

(1) How does the trend of battle affect optimal fire-support allocation? How do target priorities for fire-support systems change over the course of battle? Are they affected by the nature of combat operations (i.e., whether defensive or offensive, whether there are replacements or not, etc.)?

(2) Is optimal fire-support allocation sensitive to the nature of the target acquisition process? Will changes in target acquisition capability (in particular, new hardware developments like a laser rangefinder) necessitate changes in fire-support allocation doctrine?

(3) How should the allocation of our fire-support systems be affected by the presence of enemy fire-support systems?

Previous research review by McNicholas and Crane [24] indicated that such allocations have been largely judgmentally handled, and we wanted to establish a quantitative basis for such decisions. In particular, we wanted to show that the course of combat strongly influences the effectiveness of fire-support allocations and also that different situations require different allocation rules for maximum effectiveness (i.e., there is no "universal" fire-support allocation rule).

In Appendix B we examine the dependence of the structure of optimal time-sequential fire-support allocation policies on the quantification of military objectives by considering three specific problems, each corresponding to a different quantification of objectives (i.e., criterion functional). The three criterion functionals that we consider are as follows: (I) a weighted average of the force ratios of apposing numbers of infantry in the two infantry combat zones, (II) the difference between the total military worths (computed using linear utilities) of the surviving X and Y forces at the end of the "approach to contact," and (III) the ratio of total military worths (again computed using linear utilities) of the surviving X and Y forces. We determine the optimal time-sequential allocation of supporting fires during the "approach to contact" of friendly infantry against enemy defensive positions for each one-sided combat optimization problem. The problems are all nonconvex, and local optima are a particular difficulty in one of them. Each problem is solved, and their solutions are contrasted in order to see how the optimal fire-support allocation policy is influenced by the quantification of military objectives. Additionally, we discuss possible future research suggested by the work reported in this appendix.

The research reported in Appendix B was undertaken to determine the sensitivity of the optimal time-sequential fire-support allocation policy to the quantification of military objectives. This aspect of fire-support allocation had apparently never been quantitatively examined. The only other systematic examinations of the influences of the criterion function on the structure of optimal time-sequential fire-distribution policies known to the author are his own [34]-[43]. Furthermore, Pugh and Mayberry [31] have suggested† that an appropriate payoff, or objective function (in our terminology, criterion functional), for the quantitative evaluation of combat strategies is the loss ratio (calculated possibly using weighting factors for heterogenous forces). In Appendix B we examine to what extent these criteria are in fact equivalent.

In Appendix C we briefly consider optimal time-sequential fire-support allocation policies when there are different combat dynamics (i.e., Lanchester-type equations) in different "phases" of a battle. Such a situation occurs when a combat unit becomes "ineffective" through reduction in strength, i.e., the unit reaches its so-called "breakpoint" (see [15] and [33]). We investigate how optimal time-sequential fire-support allocation policies are modified by

†However, Pugh and Mayberry [31] do not explore the consequences of various functional forms for the criterion functional.

unit "breakpoints" being considered. The work reported in Appendix C is more exploratory than that reported in the other two appendices.

The research reported in Appendix C was undertaken to see how optimal fire-support allocation policies are affected by such models of combat unit degradation. In all the author's previous time-sequential allocation research, unit breakpoints were not considered. Furthermore, the author is not aware of any contemporary research that considers this factor. Our purpose was to see if the nature of an optimal policy is modified by such an enrichment in military detail.

6. Summary of Research Findings

Here we summarize our research results. We have (at least partially) accomplished the tasks (a) and (b) that were suggested for future research on p. 6 of our previous report [37]. Results are organized under the following headings:

(1) solution methodology for time-sequential combat problems,

(2) insights gained into optimal time-sequential fire-support allocation policies,

(3) implications for defense planning.

Items (2) and (3) differ in that the latter is a management-oriented digest of practical implications of our research, while the former is oriented towards a technical audience. Further amplification of results and conclusions is to be found in the appendices.

a. Solution Methodology for Time-Sequential Combat Problems

Our research has produced the following results on solution methodology for time-sequential combat allocation problems. Specifically, we have accomplished the following:

(1) demonstrated that judicious choice of an approximation to the combat dynamics leads to appreciable simplification in the optimal fire-support allocation policy,

(2) concluded that simplified versions of a complex problem should be initially considered in order to develop insight into the structure of the optimal policy [such simplified problems provide a point of departure for understanding more complex problems (enriched in military details)],

(3) showed that global considerations (i.e., value of criterion functional) must be used in such nonconvex optimal control problems in order to determine the optimal policy [local necessary conditions of optimality, in themselves, were inadequate to determine optimal policy],

(4) concluded that computational methods for complex problems must give consideration to structural properties of optimal policies in idealized versions like those considered in this report,

(5) illustrated how optimal control theory is applied to one-sided combat optimization problems with different combat dynamics in different "phases" of combat, denoted as "time-phased" combat [this is the first time such a model has been considered in military operations research].

b. <u>Insights Gained Into Optimal Time-Sequential Fire-Support Allocation Policies</u>

Based on our study of the optimization of time-sequential fire-support allocations using modern optimal control theory, we have reached the following conclusions:

(1) the structure of optimal time-sequential fire-support allocation policies depends on the following factors:

(a) decision criterion,
(b) combat operations model,
(c) battle termination/unit breakpoint model;

the dependence is complex; future research should concentrate on simplified models of tactical interest to explore how the optimal policy depends on these factors; research is also needed on methodology for integrating such theoretical results into practical Navy and DOD planning studies,

(2) optimal time sequential fire-support allocation policies are quite sensitive to the nature of the model of combat dynamics; based on our study of a sequence of simplified time-sequential fire-support allocation problems, we conclude that:

(a) an optimal time-sequential fire-support allocation policy depends on the dynamics of combat and target priorities evolve dynamically over the course of battle,

(b) the nature of the (Lanchester-type) target attrition process for a supporting weapon system has a major influence on the structure of the optimal fire-support policy as do those for other force interactions,

(c) the optimal time-sequential fire-support allocation policy for an attack (approach to contact) is different in structure from that for the defense of such an attack,

(d) a "linear-law" attrition process from a supporting weapon system against enemy target types may lead to supporting fires being divided between enemy targets in an optimal policy,

(e) a "square-law" attrition process always leads to concentration of fire on a single target type as the optimal policy,

(f) judicious choice (i.e., valuation in direct proportion to their rate of destroying friendly value) of the value assigned to enemy survivors (computed according to linear utilities) leads to a simple fire-support allocation policy that is also intuitively appealing; this policy remains optimal even when there are temporal variations in the effectiveness of enemy fire,

(g) simple "nearly optimal" fire-support policies may be developed through judicious approximations to the combat attrition process,

(h) if suppression is a linear function of the kill rate of the supporting weapon system, it has no effect on the optimal fire-support policy when enemy survivors are valued in direct proportion to their rate of destroying friendly value (i.e., the optimal policy is not changed if the suppressive effects are excluded from the model).

(3) optimal time-sequential fire-support policies are also quite sensitive to the criterion functional (i.e., decision criterion) chosen; based on our study of several time-sequential fire-support allocation problems (all with the same combat dynamics but different criterion functionals), we conclude that the optimal fire-support policy for a particular attack scenario is significantly influenced by the quantification of military objectives and that the most important planning decision for a side is whether it will seek to attain an "overall" or a "local" advantage from its combat operations; we found that:

   (a) the splitting of supporting fires between two enemy forces in an optimal policy (i.e., the optimality of singular subarcs) depends on whether the terminal payoff reflects the objective of attaining an "overall" military advantage or a "local" one,

   (b) switching times for changes in the ranking of target priorities are different (sometimes significantly) when the decision criterion is the difference and the ratio of the military worths (computed according to linear utilities) of total infantry survivors.

(4) optimal time-sequential fire-support allocation policies are sensitive to the modelling of unit breakpoints; the optimal policy may be significantly changed in structure by adding a nonzero force-level breakpoint into the combat model; such optimal control problems are much more difficult to solve than problems without unit breakpoints (i.e., only force-level constraints).

c. Implications for Defense Planning

In our research reported here we have studied idealizations of allocation structures that commonly occur in defense planning studies. After studying these idealizations in order to gain insight into the structure of optimal fire-support allocation strategies in the complex real-world problem, we have reached the following conclusions concerning considerations that should be brought to the attention of defense planners. These results should be kept in mind by practitioners who perform more detailed computer similation studies.

(1) The combat optimization problem should be thought of as consisting of three parts:

   (i) combatant objectives,
   (ii) conflict termination conditions,
   (iii) combat dynamics.

Optimal fire-support allocation strategies depend on all three of the above. More basic scientific research should be done on all three, particularly the first two.

(2) The time-sequential nature of target effects from fire support have a significant effect upon the optimal fire-support allocation strategies. Moreover, other combat interactions (e.g., friendly ground forces with enemy ground forces) also influence the optimal policy.

(3) It may be quite dangerous to generalize optimal fire-support strategies developed for specific problems. At present, more research is needed on specific problems in order to develop an understanding of the qualifications that may be necessary to make about specific study results.

(4) The quantification of combatant objectives does affect optimal fire-support strategies. The most important planning decision is whether to seek a "local" military advantage or an "overall" one.

(5) Unit breakpoints do affect (both directly and also indirectly) optimal fire-support strategies. More scientific work is needed on determining the relationship between unit effectiveness and unit strength.

(6) Optimal fire-support strategies must be based on ground-operations objectives. Suboptimization results when this is not done. This suboptimization may be a serious problem, since it could lead to, for example, destroying all the enemy fire-support units but losing the overall ground campaign.

7. Suggested Future Research Tasks

After performing the research documented in this report, we feel that the current state-of-the-art for applying differential-game/optimal-control theory to time-sequential combat allocation problems is such that much more significant results may be readily obtained in the future. Moreover, our previous research provides valuable perspective for identifying what appears to be the most important research tasks to be considered next. In our opinion the most important task is to continue to examine the influence of objectives on optimal fire-support strategies. Another important task is to study the structure of optimal air-war strategies.

Based on our past research experience we feel that there is much to be accomplished in the future. Specifically, we suggest the following as future research tasks:

(1) Further study of how optimal time-sequential fire-support allocation strategies depend on the nature of dynamic combat interactions. Preliminary results for a sequence of one-sided† allocation problems were given in this report (see Appendix A). We would extend these results that have already yielded important insights into optimal fire-support allocation policies. Details remain to be worked out for a number of allocation problems (see Appendix A). In particular, we would further consider the modelling of suppressive effects of supporting weapons in such work, with particular emphasis on determining how such effects influence the optimal time-sequential allocation of supporting weapons.

(2) Further study of the dependence of the structure of optimal time-sequential fire-support allocation policies on the quantification of military objectives. Based on our work documented in this report we conclude that more work needs to be done on the identification of criteria for making tactical decisions and on the quantification of such criteria. Our work indicates that the structure of optimal policies may be significantly affected by the quantification of military objectives. We would consider additional criterion functionals and would determine the corresponding optimal policy for each of these, as we have done in Appendix B (see discussion of proposed future research in Appendix B). Also, some further computational work remains to be done on the problems reported in Appendix B. In particular, we would further explore whether the loss ratio and the loss difference, i.e., the two decision criteria (see [31]), always lead to the same optimal fire-support policy.

(3) Further study of the effects on optimal time-sequential fire-support policies of a campaign composed of different "phases" (i.e., different combat dynamics in different phases of the campaign). In Appendix C we presented some preliminary results that investigate how considerations of unit "breakpoints" in the combat model affect the optimal fire-support allocation policy. We would extend these preliminary results that show that such an optimal policy is "modified near a breakpoint." Our results indicate that the nature of the planning horizon (as determined by the modelling of unit breakpoints) is a significant factor in the determination of optimal fire-support allocation strategies. In other words, otherwise appropriate results

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†Only one of the combatants is free to choose his time-sequential fire-support policy.

can be entirely misleading if unit breakpoints are incorrectly modelled (see Appendix C for further details).

(4) Study of methodology for the determination of optimal air-war strategies. This study would include the quantitative determination of optimal time-sequential allocation of aircraft to missions by application of game theory. A general framework for interfacing a simplified model with a detailed simulation (see [25]) would be developed. We would then focus on the analytic determination of optimal aircraft mission-allocation strategies by application of time-sequential game theory. A general model of combat operations would be developed (to include logistics, air and ground operations, FEBA movement, logistics, logistics interdiction, etc.), but simplified models would be studied in order to develop insights into the structure of optimal air-war strategies. A special emphasis would be placed on determining what structures for the combat dynamics lead to a saddle point in pure strategies so that the computational advantages of differential games may be exploited. We would emphasize determining how the model of combat operations influences optimal air-war strategies. This work would be based on previous studies by the author [34], [36], [37] and aided by his theoretical developments on necessary conditions of optimality for differential games (see [36], [46]). Our previous research [37] has indicated that the outcome of the ground war is a significant factor† in the determination of optimal air-war strategies and that optimal strategies developed for a model not considering the attainment of land-war objectives need not be optimal when evaluated in a model which does consider land-war objectives. Our goal would be to extend the state-of-the-art [11], [13] for such determinations.

(5) Examination of the effects of logistics constraints on optimal campaign strategies. Models would be developed to relate logistics capability to combat-effectiveness capability and then appropriate combat optimization problems formulated. Such research would provide insight into the worth of the Navy logistics (pipeline) role in combat service support missions (see Appendix E of [36]).

(6) Development of methodology for determining "good" allocation strategies (e.g., fire-distribution strategies, air-war allocation-of-aircraft strategies, etc.) in time-sequential combat games. Based on our past research we feel that it is essentially impossible to rigorously apply optimization theory to determine optimal combat strategies for realistic combat models of any appreciable complexity. However, many valuable insights into optimal combat strategies may be gained by considering simplified combat models. It would seem that "optimal" combat strategies developed for such simplified models could be used

†This is not considered, for example, in either TAC CONTENDER [47] or OPTSA I and II [7].

as a point of departure for developing "good" combat strategies for a complex combat model enriched in military detail. We would work on the development of methodology to determine "good" combat strategies (e.g., air-war strategies) by interfacing such simplified and complex models (see [25]).

REFERENCES

[1] H. Antosiewicz, "Analytic Study of War Games," Naval Res. Log. Quart. 2, 181-208 (1955).

[2] R. Bellman and S. Dreyfus, "On a Tactical Air-Warfare Model of Mengel," Opns. Res. 6, 65-78 (1958).

[3] L. Berkovitz and . Dresher, "A Game Theory Analysis of Tactical Air War," Opns. Res. 7, 599-620 (1959).

[4] L. Berkovitz and M. Dresher, "Allocation of Two Types of Aircraft in Tactical Air War: A Game Theoretic Analysis," Opns. Res. 8, 694-706 (1960).

[5] L. Berkovitz and M. Dresher, "A Multimove Infinite Game with Linear Payoff," Pacific J. Math. 10, 743-765 (1960).

[6] S. Bonder and R. Farrell (Eds.), "Development of Models for Defense Systems Planning," Report No. SRL 2147 TR 70-2 (U), Systems Research Laboratory, The University of Michigan, Ann Arbor, Michigan, Sept. 1970.

[7] J. Bracken, J. Falk, and A. Karr, "Two Models for Optimal Allocation of Aircraft Sorties," Opns. Res. 23, 979-995 (1975).

[8] C. von Clausewitz, On War, edited with an introduction by A. Rapoport, Penguin Books Ltd., Harmondsworth, Middlesex, England, 1968.

[9] M. Dresher, Games of Strategy: Theory and Applications, Prentice-Hall, Englewood Cliffs, New Jersey, 1961.

[10] D. Fulkerson and S. Johnson, "A Tactical Air Game," Opns. Res. 5, 704-712 (1957).

[11] R. Galiano and F. Miercort, "Results of a Survey of Tactical Air Campaign Models," Ketron, Inc., Arlington, Virginia, November 1974.

[12] L. Giamboni, A. Mengel, and R. Dishington, "Simplified Model of a Symmetric Tactical Air War," The RAND Corporation, RM-711, August 1951.

[13] L. Goheen, "Selection of a Method to Solve the n-Stage Game in BALFRAM," Tech. Note NWRC-TN-59, Stanford Research Institute, Menlo Park, CA, August 1975.

[14] K. Harris and L. Wegner, "Tactical Airpower in NATO Contingencies: A Joint Air-Battle/Ground-Battle Model (TALLY/TOTEM)," R-1194-PR, The RAND Corporation, Santa Monica, California, May 1974.

[15] R. Helmbold, "Decision in Battle: Breakpoint Hypotheses and Engagement Termination Data," R-772-PR, The RAND Corporation, Santa Monica, California, June 1971.

[16] Y. C. Ho, "Toward Generalized Control Theory," IEEE Trans. on Automatic Control, Vol. AC-14, 753-754 (1969).

[17] Y. C. Ho, "Differential Games, Dynamic Optimization, and Generalized Control Theory," J. Opt. Th. Appl. 6, 179-209 (1970).

[18] R. Isaacs, Differential Games, John Wiley, New York, 1965.

[19] R. Isaacs, "Differential Games: Their Scope, Nature, and Future," J. Opt. Th. Appl. 3, 283-295 (1969).

[20] Y. Kawara, "An Allocation Problem of Fire Support in Combat as a Differential Game," Opns. Res. 21, 942-951 (1973).

[21] D. Kinney, "Modeling of Weapon Suppression Effects," NWC TP 5620, Naval Weapons Center, China Lake, California, February 1974.

[22] Z. Lansdowne, G. Dantzig, R. Harvey and R. McKnight, "Development of an Algorithm to Solve Multi-Stage Games," Control Analysis Corporation, Palo Alto, California, May 1973.

[23] Lulejian & Associates, "Lulejian Artillery Model Study (DRAFT)," Lulejian & Associates, Inc. Torrance, California, June 1974.

[24] R. McNicholas and F. Crane, "Guide to Fire Support Mix Evaluation Techniques, Volume I: The Guide and Appendices A and B," Stanford Research Institute, Menlo Park, California, March 1973.

[25] R. Nolan and M. Sovereign, "A Recursive Optimization and Simulation Approach to Analysis with Application to Transportation Systems," Management Science 18, B-676 - B-690 (1972).

[26] L. Ostermann and J. Boudreau, "An Iterative Technique for Solution of Certain Multi-Move Games," Lulejian & Associates, Inc., Torrance, California, February 1972.

[27] General G. I. Pokrovsky (U.S.S.R.), Science and Technology in Contemporary War, Stevens & Sons Limited, London, 1959.

[28] Proceedings of the Conference on the State-of-the-Art of Mathematics in Combat Models, to appear.

[29] G. Pugh, "Theory of Measures of Effectiveness for General-Purpose Military Forces: Part II. Lagrange Dynamic Programming in Time-Sequential Combat Games," Opns. Res. 21, 886-906 (1973).

[30] G. Pugh and J. Mayberry, "Analysis of General-Purpose Force Procurement," Lambda Corporation, Paper 59, August 1971.

[31] G. Pugh and J. Mayberry, "Theory of Measures of Effectiveness for General-Purpose Military Forces: Part I. A Zero-Sum Payoff Appropriate for Evaluating Combat Strategies," Opns. Res. 21, 867-885 (1973).

[32] M. Schaffer, "Lanchester Models of Guerrilla Engagements," Opns. Res. 16, 457-488 (1968).

[33] S. Spring and S. Miller, "FAST-VAL: Relationships Among Casualties, Suppression, and the Performance of Company-Size Units," RM-6268-PR, The RAND Corporation, Santa Monica, California, March 1970.

[34] J. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part I," Tech. Report NPS55Tw70062A, Naval Postgraduate School, Monterey, California, June 1970.

[35] J. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part II," Tech. Report NPS55Tw72111A, Naval Postgraduate School, Monterey, California, November 1972.

[36] J. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part III," Tech. Report NPS55Tw74051, Naval Postgraduate School, Monterey, California, May 1974.

[37] J. Taylor, "Appendices C and D of 'Applications of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part III'," Tech. Report NPS55Tw74112, Naval Postgraduate School, Monterey, California, November 1974.

[38] J. Taylor, "Survey on the Optimal Control of Lanchester-Type Attrition Processes," presented at the Symposium on the State-of-the-Art of Mathematics in Combat Models, June 1973 (also Tech. Report NPS55Tw74031, Naval Postgraduate School, Monterey, California, March 1974).

[39] J. Taylor, "On the Isbell and Marlow Fire Programming Problem," Naval Res. Log. Quart. 19, 539-556 (1972).

[40] J. Taylor, "Lanchester-Type Models of Warfare and Optimal Control," Naval Res. Log. Quart. 21, 79-106 (1974).

[41] J. Taylor, "Target Selection in Lanchester Combat: Linear-Law Attrition Process," Naval Res. Log. Quart. 20, 673-697 (1973).

[42] J. Taylor, "Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent Attrition-Rate Coefficients," Naval Res. Log. Quart. 21, 683-704 (1974).

[43] J. Taylor, "On the Treatment of Force-Level Constraints in Time-Sequential Combat Problems," Naval Res. Log. Quart. 22, 617-650 (1975).

[44] J. Taylor, "Some Differential Games of Tactical Interest," Opns. Res. 22, 304-317 (1974).

[45] J. Taylor and S. Parry, "Force-Ratio Considerations for Some Lanchester-Type Models of Warfare," Opns. Res. 23, 522-533 (1975).

[46] J. Taylor, "Necessary Conditions of Optimality for a Differential Game with Bounded State Variables," IEEE Trans. on Automatic Control, Vol. AC-20, 807-808 (1975).

[47] USAF Assistant Chief of Staff, Studies and Analysis, "Methodology for Use in Measuring the Effectiveness of General Purpose Forces, SABER GRAND (ALPHA)," March 1971.

[48] H. Weiss, "Some Differential Games of Tactical Interest and the Value of a Supporting Weapon System," Opns. Res. 7, 180-196 (1959).

[49] H. Weiss, "Combat Models and Historical Data: The U.S. Civil War," Opns. Res. 14, 759-790 (1966).

APPENDIX A: Some Time-Sequential Fire-Support Allocation Problems

1. Introduction.

An important constituent part of fire support is the target allocation function which matches the specific type weapon with an acquired target within its environment.† In view of the obvious importance of fire mission allocation, it is indeed remarkable that no systematic study has apparently been made of the sensitivity of (predicted) combat outcomes to the nature of fire mission allocation techniques and/or of the quantitative justification of such allocation rules.†† Typically, these allocation rules††† are based on target priority lists along with such factors as amount of remaining ammunition, range to target, etc. Unfortunately, the target priorities appear to be judgmentally determined (the unchallengeable mystique of "military judgment" or the "quantified judgment of military experts") and not related to the dynamics of the battlefield situation. In view of this and also proposed future automation of fire direction centers (which perform the fire mission allocation function), it would appear worthwhile to develop a quantitative scientific methodology (which gives consideration to the dynamics of the battlefield situation) for the determination of fire mission allocation. In the work reported here we will develop some insights into time-sequential target priorities in fire mission allocation by the combination of optimization theory (differential game/optimal control theory) with Lanchester-type models of warfare.

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†See pp. I-33 to I-43 of [21] for a discussion of the key elements of the fire support system for systems analysis.

††Here we mean whether the allocation rules are "good" rather than whether the analyst sees the decision process in the real world this way.

†††See Table 13 on p. II-66 of [21].

Thus, the determination of optimal target allocation strategies for supporting weapon systems[†] is a major problem of military operations research. This problem is of particular relevance to the Navy mission of fire support (both by ship gunfire and by carrier-based air). The objectives of this research are to determine optimal fire-support strategies in a time-sequential fashion during the course of combat in several situations of tactical interest and to study the dependence of these strategies on the nature of the combat model. In this work consideration is given to the dynamics of combat.[††] We emphasize the development of explicit expressions for the optimal fire-support policies here. Such results are important not only for their own sake[†††] but also for testing the capabilities of proposed computational methods (for example, Lagrange dynamic programming [25] for discrete-time versions of such problems or the Lulejian successive-approximation methodology [19]).

There are many approaches to answering the question of what is the "best" allocation (over time) of supporting weapon systems. These range from operational gaming (see [23] or [39] for a discussion of terminology and background) to analytical solution of an idealized differential game. However, Berkovitz and Dresher (see p. 612 of [3]) state that "operational gaming is not a helpful device for solving a game or getting significant information about the solution." Indeed, one must distinguish between finding out how people make decisions and how they should. Most analysts agree with Berkovitz and Dresher that operational gaming is not a useful tool for answering the

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[†]See [42] for a brief discussion of the distinction between a "primary" weapon system (or infantry) and a "supporting" weapon system.

[††]This is marked contrast with essentially all the work reviewed in [21] in which apparently no consideration is given to the evolution of the course of battle.

[†††]One can clearly see the dependence of the structure of the optimal time-sequential allocation policy on model form and model parameters.

latter question. Weiss [42] has emphasized that a simplified model of a combat situation is particularly valuable when it leads to a clearer understanding of significant relationships which would tend to be obscured in a more complex (and "realistic") model. It is in this spirit that most of the work reviewed below has been done and in which we consider several simplified models for gaining insights into optimal fire-support strategies.

Our research approach is to combine Lanchester-type models of warfare with generalized control theory (i.e. optimization theory for dynamic systems). This research program is described in more detail elsewhere [34], [35]. In the initial phase of research reported here we will examine a sequence of one-sided† time-sequential allocation problems in order to study the dependence of the optimal fire-support policy on the nature of the combat attrition process. These problems represent various versions of the same basic fire-support situation. Many of these optimal control problems are solved in detail, and such results are useful for evaluating computational algorithms (see above). In future work, we would extend these results to two-sided optimization problems (i.e. time-sequential combat games).††

2. Research Objectives.

The general objective of this research is to develop combat attrition models and optimization techniques to extend the state-of-the-art for the determination of optimal time-sequential fire-support strategies in various tactical scenarios. The specific objectives of the initial phase of this research are to determine optimal time-sequential fire-support policies in

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†In other words, only one of the combatants is free to choose his time-sequential fire-support allocation policy.

††R. Isaacs [12] has emphasized the difficulties attendant with the transition from one-sided to two-sided dynamic optimization problems.

several situations of tactical interest and to study the dependence of these policies on the functional form of the model of combat dynamics.

3. Review of Previous Work.

The determination of optimal target allocation strategies for supporting weapon systems is in one form or another probably one of the most extensively studied problems in both the open literature and also classified sources. During World War II the problem of the appropriate mixture of tactical and strategic forces (another aspect of the optimal fire-support strategy problem) was extensively debated by experts. Some analysis details are to be found in the classic book by Morse and Kimball (see pp. 73-77 of [22]). The problem was studied at RAND in the late 1940's and early 1950's (see [10]) and elsewhere (see [1]). It would probably not be too far-fetched to say that this problem stimulated early research on both dynamic programming (see [2]) and also differential games (see [10], [11]). Today the problem of optimal air-war strategies is being extensively studied by a number of organizations (see, for example, [26], [40]).

The most widely studied Lanchester-type differential game† has been A. Mengel's "War of Attrition and Attack" (see pp. 96-105 of Isaacs' book [11]) (also see [35], [36]). Optimal time-sequential "air-war" strategies for two versions of this problem are developed in Isaacs' now classic book [11]. Discrete-time versions of this problem have been considered by Berkovitz and Dresher [3], [4] (see Appendix D of [37] for further references to other related work). Another related problem was considered by Weiss [42], who studied the optimal selection of targets for engagement by a supporting weapon system.††

†This term was apparently first coined in [34] (see also [35]).

††See [33], however, for a justification of the optimality of strategies determined by Weiss [42]. A general solution algorithm is also presented in this paper [33].

Kawara [15] has studied optimal strategies for supporting weapon systems in an attack scenario which is a variation of the model considered by Weiss [42]. Kawara [15] concludes that each side's optimal strategy for the distribution of its supporting weapon system's fire is to always concentrate all fire on the enemy's supporting weapon system (counter-battery fire) during the early stages of battle (if the total prescribed length of battle is long enough) and then later to switch to concentration of all fire on the enemy's infantry. He states that this switching time "does not depend on the current strength of either side but only on the effectivenesses of both sides' supporting units" (p. 951 of [15]). Moreover, an optimal time-sequential fire-support strategy has the property of always requiring concentration of supporting fires on enemy infantry during the final stages of battle.

It is shown in [37], however, that Kawara [15] considered essentially the only type of objective function which yields the switching times (i.e. times of change from counter-battery fire to counter-infantry fire) to be independent of force levels in the optimal time-sequential strategies. Additionally, we showed that for other attrition structures the optimal fire-distribution strategy could consist of splitting one's fire between enemy infantry and artillery (counter-battery fire). Thus, Kawara's conclusions about the nature of optimal time-sequential fire-support strategies are not of general applicability, and one must determine optimal fire-support strategies on a problem by problem basis. Other time sequential fire-support allocation problems have been considered in our past work [35], [36], [37]. Other recent work has considered various conceptual and computational aspects of time-sequential combat games [25], [26], [27].

On a much more applied level, McNicholas and Crane [21] report the results of research to identify a comprehensive methodology for evaluating fire-support mixes. This is apparently the most comprehensive piece of applied work on fire-support evaluation methodology and contains analysis of the fire-support function and process, including a summary of previous work and procedures for fire mission allocation. This problem (i.e. fire mission allocation) is treated mainly in a descriptive manner (i.e. how do fire control officers actually select weapons rather than how should they) with apparently no consideration given to time-sequential aspects.

Lulejian and Associates, Inc. [19] report methodology for the optimum allocation of field artillery fires over time between counter-fire and other forms of fire support to engaged troops. The dynamics of combat are modelled by "interactive equations" with time treated discretely. A very credible brigade-level model of conventional tactical combat is developed. The Lulejian work is similar in concept to the work reported here, only it is much more detailed and complex. However, the computational optimization algorithm (referred to as an enforceable-bound technique which is basically a successive approximation method) apparently has no mathematical justification and has not been reported in the open literature.

4. A General (One-Sided) Time-Sequential Fire-Support Problem.

In this section we consider a general one-sided† time-sequential fire-support allocation problem. Particularizations of this general problem will be subsequently considered in sections below.

Let us consider heterogeneous X forces in combat against heterogeneous Y forces along a "front." Each side is composed of primary units (or infantry)

†In other words only one of the combatants is free to choose his time-sequential fire-support policy.

and fire-support units. The X infantry (denoted as X_1 and X_2) is in direct combat against the Y infantry (denoted as Y_1 and Y_2). We may consider X_1 and X_2 to be two different infantry units operating on spatially separated pieces of terrain. We assume that the X_1 infantry unit is in combat against the Y_1 infantry unit and similarly for X_2 and Y_2 with no "crossfire" (i.e. the X_1 infantry is not attrited by the Y_2 infantry).

For the battle described above, we will consider only the "approach to contact" phase of the battle. We assume that one force attacks the other along the "front." In most of the particularizations considered below the attacker will be the X force. In this case the "approach to contact" phase of battle is the time from the initiation of the advance of the X_1 and X_2 forces towards the Y_1 and Y_2 defensive positions until the X_1 and X_2 forces actually make contact (assumed to be simultaneous in the two combat areas) with the enemy infantry in "hand-to-hand" combat. It is assumed that this time is fixed and known to X.

The associated Lanchester attrition-rate coefficients for combat between the X_i force and the Y_i force are denoted as $a_i(t,x_i)$ for the effectiveness of Y_i's fire and as $b_i(t,y_i)$ for the effectiveness of X_i's fire. Based upon different sets of assumptions† as to the conditions under which the combat attrition process takes place, these coefficients (e.g. $a_i(t,x_i)$) take different functional forms. We will give such assumptions for each of the problems considered subsequently below. Additionally, both forces may receive replacements†† continuously over time with the corresponding

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†The most comprehensive compendium of sets of assumptions and corresponding Lanchester-type attrition processes known to this author is the IDA report by A. Karr [14] (see also [5], [41]).

††Alternatively, we may consider that additional forces are entering into the enemy's "field of fire" at these rates.

replacement rates being denoted as $r_i(t)$ for the X_i forces and as $s_i(t)$ for the Y_i forces.

The Y forces may be supported by fire-support units (denoted as Z with force level denoted as $z(t)$) which are invulnerable to (i.e. out of range of) the direct fires of the X forces. In general, these fire-support units cause attrition to the X_i forces at a rate $A_i(t,x_i)$. (It should be noted that this coefficient includes an allocation factor for the distribution of Z fire over the X forces.) Particularizations of these general loss rates and the circumstances under which they are hypothesized to apply are given in the specific problems considered below. For the investigation reported here, we assume that these Z fire-support units do not engage the W fire-support units.

During the "approach to contact" the fire-support units of the X force (denoted as W) distribute their fire over the Y forces and (if present) their fire-support units (i.e. Z). The purpose of this investigation is to determine the best possible such time-sequential allocation according to a given criterion (given below). Let ϕ_i denote the fraction of W fire-support units which fire at Y_i and $B_i(t,y_i)$ denote the Lanchester attrition-rate coefficient corresponding to the effectiveness of this fire against the Y_i forces.† Again, particularizations of these general loss rates are considered in the specific problems below. Further, let ϕ denote the fraction of the W fire-support units which fire at the enemy Z fire-support units (if these are present in the specific model under consideration) and $B(t,z)$ denote the corresponding general Lanchester attrition-rate coefficient. We then have that

†Thus, the loss rate for the Y_i forces as a result of these supporting fires is the product of these two factors, i.e. (loss rate for Y_i from W fire support) = $\phi_i B_i(t,y_i)$.

$\phi + \sum_{i=1}^{2} \phi_i = 1$ (so that ϕ may be eliminated for convenience). Since the W fire-support units are not in the combat zone (and consequently do not suffer attrition from the Y forces) and are not engaged by the enemy's Z fire-support units,[†] for constant ϕ_i there is a constant number of fire-support units firing at Y_i. The combat situation described above is shown diagrammatically in Figure 1.

It is the objective of the X force to utilize its fire-support units over time in such a manner so as to achieve the "most favorable" situation measured in terms of the net worth of survivors (computed according to linear utilities)[††] at the end of the "approach to contact" at which time the force separation between opposing infantries is zero and supporting fires must be lifted from the enemy's infantry positions in order not to also kill friendly forces. Thus, we have the following optimal control problem for the determination of the optimal time-sequential fire-support allocation policy (denoted as $\phi_i^*(t)$ for $0 \leq t \leq T$ (with $i = 1,2$), where T denotes the time of the end of the "approach to contact") for the W fire-support units.

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T) \right\},$$

with stopping rule: $$t_f - T = 0,$$

subject to: (battle dynamics)

$$\frac{dx_i}{dt} = -a_i(t,x_i)y_i - A_i(t,x_i)z + r_i(t),$$

$$\frac{dy_i}{dt} = -b_i(t,y_i)x_i - \phi_i B_i(t,y_i) + s_i(t) \quad \text{for } i = 1,2,$$

$$\frac{dz}{dt} = -(1 - \sum_{i=1}^{2} \phi_i)B(t,z), \tag{1}$$

[†]This is not an essential assumption. The analysis presented here is easily extended to the case in which the Z fire-support units engage in counter-battery fire.

[††]Other criterion functionals are considered in Appendix B.

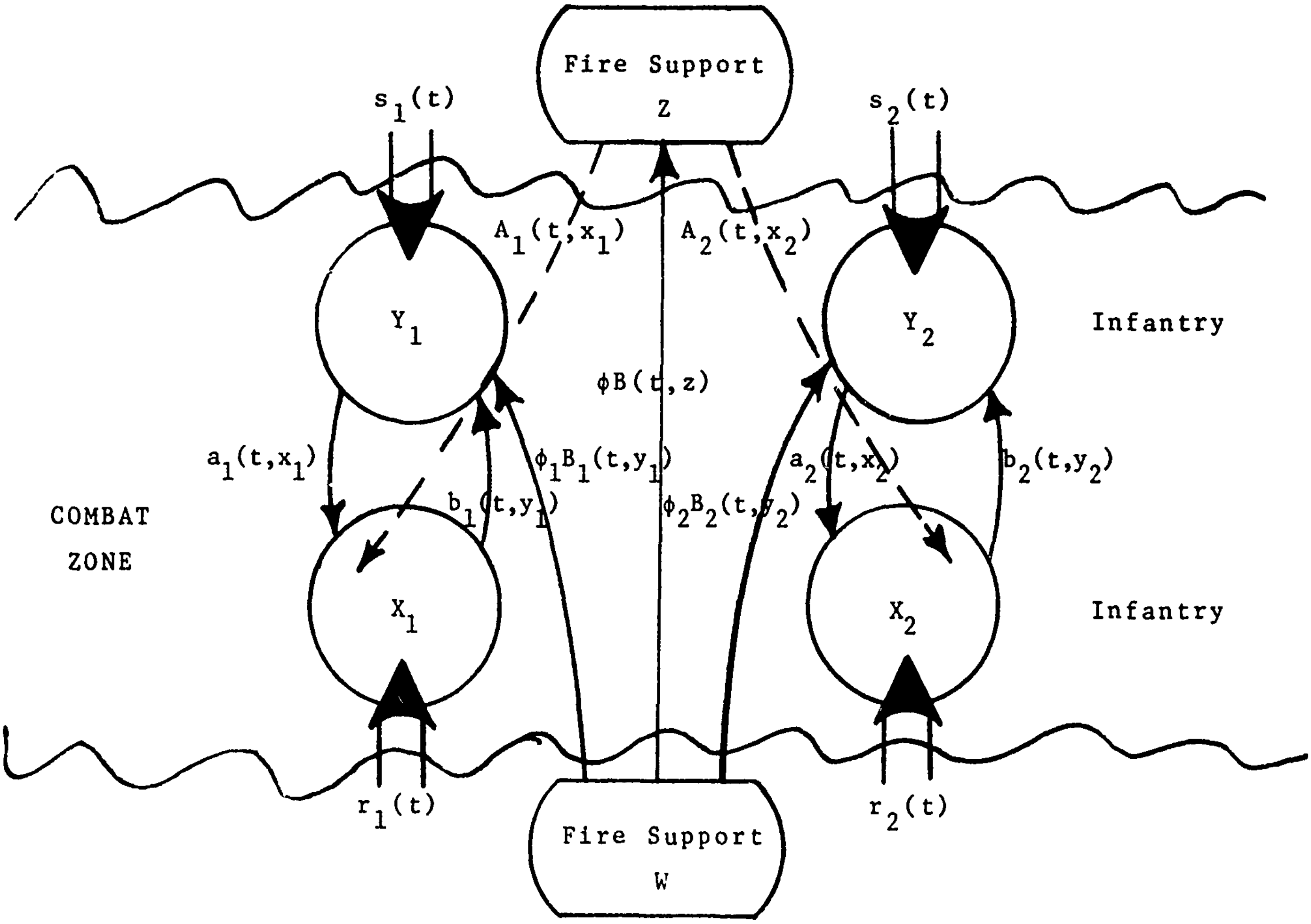

Figure 1. Diagram of General (One-Sided) Time-Sequential Fire-Support Problem Faced by W Fire-Support Units.

with initial conditions

$$x_i(t=0) = x_i^o \text{ and } y_i(t=0) = y_i^o \text{ for } i = 1,2 \text{ and } z(t=0) = z_0,$$

and

$x_1, x_2, y_1, y_2, z \geq 0$ (State Variable Inequality Constraints)

$\phi_1 + \phi_2 \leq 1$ and $\phi_i \geq 0$ for $i = 1,2$ (Control Variable Inequality Constraints),

where

v_i denotes the value (utility) per unit of surviving X_i force, similarly for w_i (which corresponds to the Y_i force),

$x_i(t)$ denotes the number of X_i infantry at time t, similarly for $y_i(t)$,

$z(t)$ denotes the number of Z fire-support units at time t,

$a_i(t,x_i)$ is a (Lanchester) attrition-rate coefficient (reflecting the effectiveness of Y_i fire against X_i), similarly for $b_i(t,y_i)$,

$A_i(t,x_i)$ is a (Lanchester) attrition-rate coefficient (reflecting the effectiveness of Z supporting fires against X_i), similarly for $B_i(t,y_i)$ and $B(t,z)$,

t_f (with numerical value T) denotes the end of the optimal control problem,

and ϕ_i denotes the fraction of W fire support directed at Y_i.

In the ensuing analysis of specific problems we will only consider cases in which no force level is driven to zero (i.e. $x_i, y_i, z > 0$). In Appendix C we consider some models in which this assumption is relaxed and breakpoints are considered for the various forces. Unfortunately, this leads to quite complex mathematical details.

5. A Sequence of Problems to Study the Effects of Nature of Attrition Structure on the Optimal Fire-Support Policy.

The effects of the nature of the attrition structure on the optimal fire-support policy are studied by examining a sequence of specific problems and then contrasting the structures of the optimal time-sequential fire-support policies for these problems. In this manner we will study the dependence of these policies on the functional form of the model of combat dynamics. The problems that have been considered are summarized in Table I. In most cases the optimal time-sequential fire-support policy will be determined by the mathematical theory of optimal control (see [6], [24]).

6.1. Problem 1.

In this section we will consider the special case of the general problem (1) graphically depicted in Figure 1 in which both infantries in each of the two combat zones cause attrition to the enemy forces at rates proportional to only the numbers of firers. The corresponding Lanchester attrition-rate coefficients are for mathematical convenience assumed to be constant over the course of battle. It is convenient to refer to the attrition of a target type as being a "square-law" process when the casualty rate is proportional to only the number of enemy firers and as being a "linear-law" process when it is proportional to the product of the numbers of enemy firers and remaining targets. Considering both the work of Brackney [5] and also that of Karr [14], we see that one set of conditions under which a "square-law" attrition process occurs[†] is when "aimed" fire is used and a constant time (indpendent of the target type force level) is required to acquire targets (see [5], [14], and [41] for a further discussion of such assumptions and alternative sets of

[†]To be precise, one can only conjecture that such an attrition process probably occurs under the stated conditions.

TABLE I

Summary of Problems Considered to Study Effects of
Nature of Attrition Structure on Optimal Fire- Support Policy

PROBLEM	$a_i(t,x_i)$	$b_i(t,y_i)$	$A_i(t,x_i)$	$B_i(t,y_i)$	$B(t,z)$	$r_i(t)$	$s_i(t)$
1	$\bar{a}_i$	$\bar{b}_i$	0	$c_i y_i$	0	0	0
2	$\bar{a}_i$	0	0	$c_i y_i$	0	$r_i(t)$	0
2a	$\bar{a}_i$	0	0	$c_i y_i$	0	0	0
2b	$a_i(t)$	0	0	$c_i y_i$	0	0	0
3	0	$\bar{b}_i$	0	$c_i y_i$	0	0	$\bar{s}_i$
4	$\bar{a}_i$	$\bar{b}_i y_i$	0	$c_i y_i$	0	0	0
5	$\bar{a}_i$	$\bar{b}_i$	α_i	$c_i y_i$	β	0	0
6	$\bar{a}_i$	0	α_i	$c_i y_i$	β	$r_i(t)$	0
7	$\bar{a}_i$	0	$\alpha_i x_i$	$c_i y_i$	β	0	0
8	$\bar{a}_i$	0	$\alpha_i x_i$	$c_i y_i$	β_z	$r_i(t)$	0
9	$\bar{a}_i$	$\bar{b}_i$	0	c_i	0	0	0
10	$\bar{a}_i$	0	0	c_i	0	$r_i(t)$	$\bar{s}_i$

conditions which lead to such an attrition process). The X fire-support units (denoted as W) deliver "area fire" against the Y_i forces.† In this case, the Y_i attrition rate is proportional to the Y_i force level (see [41]; also [14]). Other portions of the general model (1) depicted in Figure 1 are assumed to be absent.

In other words, we will consider the case in which the following hold:

$$a_i(t,x_i) = \bar{a}_i = \text{constant},$$
$$b_i(t,y_i) = \bar{b}_i = \text{constant},$$
$$B_i(t,y_i) = c_i y_i \text{ where } c_i \text{ is constant},$$
$$\text{and } A_i(t,x_i) = B(t,z) = r_i(t) = s_i(t) = 0.$$

For notational convenience we will again denote $\bar{a}_i$ as a_i, etc. The particular combat situation is shown diagrammatically in Figure 2. It is then convenient to re-state the problem as follows:

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T) \right\},$$

with stopping rule: $t_f - T = 0$,

$$\text{subject to: } \frac{dx_i}{dt} = -a_i y_i, \tag{2}$$

$$\frac{dy_i}{dt} = -b_i x_i - \phi_i c_i y_i \quad \text{for } i = 1,2,$$

with initial conditions

$$x_i(t=0) = x_i^o \quad \text{and} \quad y_i(t=0) = y_i^o \quad \text{for} \quad i = 1,2,$$

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†In other words, we assume that X's fire support units fire (at a constant rate) into the (constant) are containing the enemy's infantry without feedback as to the destructiveness of this fire.

We will consider only the first of these here.

CASE I: $w_1/(a_1v_1) = w_2/(a_2v_2)$; i.e. $w_i = ka_iv_i$ for $i = 1,2$.

In this case enemy survivors are valued in direct proportion to the rate in which they destroy value of the friendly forces. We then have

$$S_\phi(\tau=0) = k(a_1c_1v_1y_1^f - a_2c_2v_2y_2^f) \tag{24}$$

and

$$\dot{S}_\phi(\tau=0) = a_1c_1v_1y_1^f - a_2c_2v_2y_2^f + k(a_1b_1c_1v_1x_1^f - a_2b_2c_2v_2x_2^f). \tag{25}$$

There are now three further subcases: (A) $a_1b_1 = a_2b_2$, (B) $a_1b_1 > a_2b_2$, and (C) $a_1b_1 < a_2b_2$. We will consider only the first of these here.

SUBCASE A: $a_1b_1 = a_2b_2$ (and $w_i = ka_iv_i$ for $i = 1,2$).

We will focus on conditions which must necessarily hold on a singular subarc. In this case the singular control (19) becomes

$$\phi_S = (c_2^2R_2 - 2a_1b_1(c_1Q_1 - c_2Q_2))/(c_1^2R_1 + c_2^2R_2). \tag{26}$$

From (12) it follows that $\frac{d}{d\tau}(c_1Q_1 - c_2Q_2) = -2(c_1S_1 - c_2S_2)$, whence by (18) on a singular subarc

$$c_1Q_1 = c_2Q_2 + \text{constant}. \tag{27}$$

It is convenient to consider the force ratio $r_i = x_i/y_1$, and then by (2) the force ratio statisfies the following Riccate equation

$$\dot{r}_i = a_i - \phi_i c_i r_i - b_i r_i^2. \tag{28}$$

We also compute that

$$\frac{d}{d\tau}\left(\frac{r_1}{r_2}\right) = \left\{a_1b_1\left(\frac{1}{b_1r_1} - \frac{1}{b_2r_2}\right) - (b_1r_1 - b_2r_2) - \phi c_1 + (1-\phi)c_2\right\}\left(\frac{r_1}{r_2}\right), \tag{29}$$

and

$$x_1, x_2, y_1, y_2 \geq 0 \quad \text{(State Variable Inequality Constraints)}$$

$$\phi_1 + \phi_2 = 1 \quad \text{and} \quad \phi_i \geq 0 \quad \text{for} \quad i = 1,2$$

$$\text{(Control Variable Inequality Constraints)},$$

where all symbols are as defined above. It will be convenient to consider the single control variable ϕ defined by

$$\phi = \phi_1 \quad \text{so that} \quad \phi_2 = (1-\phi) \quad \text{and} \quad 0 \leq \phi \leq 1. \tag{3}$$

In the analysis presented here we assume that no force level ever becomes zero (i.e. $x_i, y_i > 0$ always). In Appendix C we consider some models in which this assumption is relaxed and breakpoints are considered for the various forces.

6.1.1 Necessary Conditions of Optimality.

We characterize an optimal fire-support policy by application of modern optimal control theory. The Hamiltonian [6] is given by (using (3))

$$H = -\sum_{i=1}^{2} p_i a_i y_i - q_1(b_1 x_1 + \phi c_1 y_1) - q_2(b_2 x_2 + (1-\phi) c_2 y_2), \tag{4}$$

so that the maximum principle yields the extreme control law

$$\phi^*(t) = \begin{cases} 1 & \text{for } S_\phi(t) > 0, \\ 0 & \text{for } S_\phi(t) < 0, \end{cases} \tag{5}$$

where $S_\phi(t)$ denotes the ϕ-switching function defined by

$$S_\phi(t) = c_1(-q_1)y_1 - c_2(-q_2)y_2. \tag{6}$$

The adjoint system of equations for the dual variables (again using (3) for convenience) is given by (assuming that $x_i(T), y_i(T) > 0$)

$$\dot{p}_i = b_i q_i \qquad \text{with } p_i(T) = v_i,$$

and

$$\dot{q}_i = a_i p_i + \phi_i^* c_i q_i \qquad \text{with } q_i(T) = -w_i \quad \text{for} \quad i = 1,2. \tag{7}$$

Computing the first two time derivatives of the switching function

$$\dot{S}_\phi(t) = c_1(b_1q_1x_1-a_1p_1y_1) - c_2(b_2q_2x_2-a_2p_2y_2), \tag{8}$$

$$\begin{aligned}\ddot{S}_\phi(t) &= 2a_1b_1c_1(p_1x_1-q_1y_1) - 2a_2b_2c_2(p_2x_2-q_2y_2)\\ &+ \phi c_1^2(b_1q_1x_1+a_1p_1y_1) - (1-\phi)c_2^2(b_2q_2x_2+a_2p_2y_2),\end{aligned} \tag{9}$$

we find that it is convenient for purposes of synthesizing extremals† to introduce the variables P_i, Q_i, R_i, and S_i for $i = 1,2$ defined as follows:

$$\begin{aligned} P_i &= p_ix_i + q_iy_i, & Q_i &= p_ix_i - q_iy_i,\\ R_i &= b_iq_ix_i + a_ip_iy_i, & S_i &= b_iq_ix_i - a_ip_iy_i,\end{aligned} \tag{10}$$

and then (using (3)) we have for $i = 1,2$

$$P_i(t) = v_ix_i^f - w_iy_i^f = \text{constant}, \tag{11}$$

$$\dot{Q}_i = 2S_i \qquad \text{with} \qquad Q_i(T) = v_1x_1^f + w_1y_1^f, \tag{12}$$

$$\dot{R}_i = \phi_ic_iS_i \qquad \text{with} \qquad R_i(T) = -b_iw_ix_i^f + a_iv_iy_i^f, \tag{13}$$

$$\dot{S}_i = 2a_ib_iQ_i + \phi_ic_iR_i \qquad \text{with} \qquad S_i(T) = -b_iw_ix_i^f - a_iv_iy_i^f < 0, \tag{14}$$

where x_i^f denotes $x_i(T)$ and similarly for y_i^f. The first two time derivatives of the switching function may then be written as

$$\dot{S}_\phi(t) = c_1S_1 - c_2S_2, \tag{15}$$

$$\begin{aligned}\ddot{S}_\phi(t) &= 2a_1b_1c_1Q_1 - 2a_2b_2c_2Q_2\\ &+ \phi c_1^2R_1 - (1-\phi)c_2^2R_2.\end{aligned} \tag{16}$$

Thus, we see that on a singular subarc†† we have [6], [16]

$$c_1(-q_1)y_1 = c_2(-q_2)y_2, \tag{17}$$

$$c_1S_1 = c_2S_2, \tag{18}$$

---

†By an extremal we mean a trajectory on which the necessary conditions of optimality are satisfied.

††See [31] for a further discussion.

with the singular control given by

$$\phi_S = (c_2^2R_2-2(a_1b_1c_1Q_1-a_2b_2c_2Q_2))/(c_1^2R_1+c_2^2R_2). \tag{19}$$

On such a singular subarc we require that

$$c_1^2R_1 + c_2^2R_2 \geq 0, \tag{20}$$

in order that the generalized Legendre-Clebsch condition be satisfied, since

$$\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = c_1^2R_1 + c_2^2R_2.$$

6.1.2 Synthesis of Extremals.

In this section we will partially synthesize the extremal fire-support policy† in one special case. In synthesizing extremals by the usual backwards construction procedure (see, for example, [29] or [31]) it is convenient to introduce the "backwards" time τ defined by $\tau = T-t$. We then have

$$S_\phi(\tau=0) = a_2c_2v_2y_2^f\left(\frac{w_1}{a_1v_1}\right)\left\{\frac{a_1c_1v_1y_1^f}{a_2c_2v_2y_2^f} - \left(\frac{w_2}{a_2v_2}\right)\Bigg/\left(\frac{w_1}{a_1v_1}\right)\right\} \tag{21}$$

$$\overset{\circ}{S}_\phi(\tau) = -c_1S_1 + c_2S_2, \tag{22}$$

and

$$\overset{\circ\circ}{S}_\phi(\tau) = 2a_1b_1(c_1Q_1) - 2a_2b_2(c_2Q_2) + \phi c_1^2R_1 - (1-\phi)c_2^2R_2, \tag{23}$$

where $\overset{\circ}{S}_\phi$ denotes the "backwards" time derivative $\overset{\circ}{S}_\phi = dS_\phi/d\tau$. Without loss of generality we may assume that $w_1/(a_1v_1) \geq w_2/(a_2v_2)$, and then ther are two cases to be considered:

(I) $w_1/(a_1v_1) = w_2/(a_2v_2)$,

(II) $w_1/(a_1v_1) > w_2/(a_2v_2)$.

---

†By an extremal policy we mean one for which the necessary conditions of optimality are satisfied. It may, of course, not turn out to be an optimal policy.

We will consider only the first of these here.

CASE I: $w_1/(a_1 v_1) = w_2/(a_2 v_2)$; i.e. $w_i = k a_i v_i$ for $i = 1,2$.

In this case enemy survivors are valued in direct proportion to the rate in which they destroy value of the friendly forces. We then have

$$S_\phi(\tau=0) = k(a_1 c_1 v_1 y_1^f - a_2 c_2 v_2 y_2^f) \tag{24}$$

and

$$\dot{S}_\phi(\tau=0) = a_1 c_1 v_1 y_1^f - a_2 c_2 v_2 y_2^f + k(a_1 b_1 c_1 v_1 x_1^f - a_2 b_2 c_2 v_2 x_2^f). \tag{25}$$

There are now three further subcases: (A) $a_1 b_1 = a_2 b_2$, (B) $a_1 b_1 > a_2 b_2$, and (C) $a_1 b_1 < a_2 b_2$. We will consider only the first of these here.

SUBCASE A: $a_1 b_1 = a_2 b_2$ (and $w_i = k a_i v_i$ for $i = 1,2$).

We will focus on conditions which must necessarily hold on a singular subarc. In this case the singular control (19) becomes

$$\phi_S = (c_2^2 R_2 - 2a_1 b_1 (c_1 Q_1 - c_2 Q_2))/(c_1^2 R_1 + c_2^2 R_2). \tag{26}$$

From (12) it follows that $\frac{d}{d\tau}(c_1 Q_1 - c_2 Q_2) = -2(c_1 S_1 - c_2 S_2)$, whence by (18) on a singular subarc

$$c_1 Q_1 = c_2 Q_2 + \text{constant}. \tag{27}$$

It is convenient to consider the force ratio $r_i = x_i/y_i$, and then by (2) the force ratio statisfies the following Riccate equation

$$\dot{r}_i = a_i - \phi_i c_i r_i - b_i r_i^2. \tag{28}$$

We also compute that

$$\frac{d}{d\tau}\left(\frac{r_1}{r_2}\right) = \left\{a_1 b_1 \left(\frac{1}{b_1 r_1} - \frac{1}{b_2 r_2}\right) - (b_1 r_1 - b_2 r_2) - \phi c_1 + (1-\phi) c_2\right\}\left(\frac{r_1}{r_2}\right), \tag{29}$$

since for $a_1b_1 = a_2b_2$

$$r_1/a_1 = r_2/a_2 \Longleftrightarrow b_1r_1 = b_2r_2. \tag{30}$$

On a <u>singular subarc</u> where $0 < \phi_S < 1$ we have via (13) that

$$dR_1/dR_2 = \phi_S/(1-\phi_S). \tag{31}$$

If we wind up on a singular subarc (for a finite interval of time) at the end of the "approach to contact" then via (24) and (25) we have

$$a_1c_1v_1y_1^f = a_2c_2v_2y_2^f \quad \text{and} \quad a_1b_1c_1v_1x_1^f = a_2b_2c_2v_2x_2^f, \tag{32}$$

whence

$$b_1r_1^f = b_2r_2^f \quad \text{or, equivalently,} \quad r_1^f/a_1 = r_2^f/a_2. \tag{33}$$

From (20) (using (13) and (32))we find that it is necessary that

$$y_i^f \geq kb_ix_i^f. \tag{34}$$

By (12) and (27), we have

$$c_1Q_1 = c_2Q_2, \tag{35}$$

whence by (26)

$$\phi_S = c_2^2R_2/(c_1^2R_1+c_2^2R_2). \tag{36}$$

Considering (13), (31), and (37), we find that

$$c_1R_1 = c_2R_2, \tag{37}$$

whence by (36)

$$\phi_S = c_2/(c_1+c_2), \tag{38}$$

so that (29), (33), (38), and the uniqueness of solution to the Riccati equation (28) yield

$$b_1r_1 = b_2r_2 \quad \underline{\text{on a singular subarc}}. \tag{39}$$

Thus, we have shown that if we wind up on a singular subarc for a finite interval of time ending at t = T, then on the singular subarc (39) holds. In this case (29) superfluously yields $\frac{d}{d\tau}\left(\frac{r_1}{r_2}\right) = 0$.

6.1.3. A Consequence of Fire-Support.

In the work at hand we examine optimal fire-support policies under the assumption that $x_i, y_i > 0$. Another aspect that we will briefly discuss here (but not at this time pursue further) is the quantification of how the application of fire support changes the course of combat. This may be quantitatively seen by consideration of the force-ratio equation

$$dr_i/dt = b_i r_i^2 + \phi_i(t) c_i r_i - a_i, \tag{40}$$

where $r_i = x_i/y_i$. Let us consider a battle between the X_i and Y_i forces which terminates at the first time that either of two given "breakpoint" force ratios is reached. These "breakpoint" force ratios, denoted as $r^f_{X_i}$ when X_i wins and as $r^f_{Y_i}$ when Y_i wins, satisfy $0 \le r^f_{Y_i} < r^o_i < r^f_{X_i} \le +\infty$. Corresponding to a fight until the annihilation of one side or the other is the case in which $r^f_{Y_i} = 0$ and $r^f_{X_i} = +\infty$. For mathematical convenience we will consider this special case, with results being readily extended to the general case. As noted in [34], the entire topic of modelling battle termination is a problem area in contemporary defense planning studies, and there is far from universal agreement on this topic.

Let us now see how the force-ratio equation (40) can help us to quantitatively evaluate the effect of fire support on battle outcome. We observe that for a fight-to-the-finish that (a) X_i wins at t = T when $r_i(T) = +\infty$, and (b) Y_i wins when $r_i(T) = 0$. Thus, it seems appropriate to say that "the course of battle is moving towards an X_i victory" when $dr_i/dt > 0$ (or, simply, that "X_i is winning"). Moreover, $dr_i/dt > 0$ if and only if

$$b_i x_i^2(t) + \phi_i(t) c_i x_i(t) y_i(t) > a_i y_i^2(t), \qquad (41)$$

which may be considered to be a "local" condition for X_i to win. When $\phi_i(t)$ is constant over the course of battle, then (41) holding at $t = 0$ is a necessary and sufficient condition for X_i to win. This may be proven by considering $r_i^+(t) = \{-\phi_i(t)c_i + \sqrt{\phi_i^2(t)c_i^2+4a_ib_i}\}/2b_i$ and $r_i^-(t) = \{-\phi_i(t)c_i - \sqrt{\phi_i^2(t)c_i^2+4a_ib_i}\}/2b_i$. It follows that $dr_i/dt < 0$ for $r_i^- < r_i < r_i^+$ and that $r_i^- < 0 < r_i^+$. Thus, $r_i^+(t)$ being nonincreasing and $dr_i/dt(t=0) > 0$ is a sufficient condition for X_i to win. We also observe that $\partial r_i^+/\partial\phi_i < 0$ always so that X can trade off initial infantry strength x_i with fire support in attempting to prevail in combat.

In other words, consideration of (41) shows us quantitatively how the application of fire support may change the course of combat (in the sense of determination of the victor). We will not pursue such matters further here, however.

6.1.4. Need for Approximations.

We have not fully determined the optimal fire-support policy for Problem 1. As seen in Sections 6.1.1 and 6.1.2 above the details are quite complex. Moreover, the optimal policy (expressed as a closed-loop control (see [34], [38])) depends on the five state variables t, x_1, x_2, y_1, and y_2. Thus, $\phi^* = \phi^*(t,\underset{\sim}{x},\underset{\sim}{y})$. In the future we plan to determine the optimal policy in some special cases (e.g. $w_i = ka_iv_i$ and $a_1b_1 = a_2b_2$). Even so, it will be quite complex to describe because of the dimensionality of the state space.

It is therefore of interest to use approximations to simplify solution details. There are two limiting cases of the battle (2) that may be considered in this respect, depending on whether X is the attacker or the defender. For these limiting cases, the approximations are

(1) X attacks: $b_i = 0$,

(2) Y attacks: $a_i = 0$.

We will see that such approximations (for example, that when X attacks, the attrition caused by the attacking forces during the "approach to contact" is negligible, i.e. $b_i = 0$ (see [37])) lead to a considerable simplification of analytic solution details and then the optimal fire-support policy may be readily determined. Without the use of approximations, it appears to be impossible to develop insights into the optimal fire-support policy for (2) without computational studies. Furthermore, the availability of complete analytic solutions for simplified versions of (2) (i.e. when the above approximations are used) is useful for checking the adequacy of proposed computational algorithms (see [6], [25], [26]).

6.2. Problem 2.

In this section we will consider a version† (see Section 6.1.4) of Problem 1 as given by (2) in which heterogeneous X forces attack the static defense of heterogeneous Y forces along a "front." We assume that the Y_i forces cause attrition to the X_i forces according to a "square-law" attrition process. [As Brackney has pointed out [5], one would expect such a process to occur when the time to acquire targets is negligible in comparison with the time

---

†This is, of course, a special case of the general problem (1) (see Section 4) graphically depicted in Figure 1 in which the following hold:

$$\begin{aligned} a_i(t,x_i) &= \bar{a}_i = \text{constant}, \\ b_i(t,y_i) &= \bar{b}_i = \text{constant}, \\ B_i(t,y_i) &= c_i y_i \text{ where } c_i \text{ is constant}, \\ r_i(t) &\text{ is a given piecewise continuous function} \\ \text{and } A_i(t,x_i) &= B(t,z) = s_i(t) = 0. \end{aligned}$$

For notational convenience we will again denote $\bar{a}_i$ as a_i, etc.

to destroy them. Such a situation is to be expected when one force assaults another (see [5]).] The attrition of the Y_i force by the attacking X_i is assumed to be negligible. [We assume that the objective of the X_i forces during the "approach to contact" is to close with the enemy position as rapidly as possible so that small arms fire by X_i is held to a minimum or that firing is done "on the move."] As before, the X fire-support units (denoted as W) deliver "area fire" against the Y_i forces† (see [28] for a discussion of the determination of the Lanchester attrition-rate coefficient for the fire-support units). All Lanchester attrition-rate coefficients are assumed to be constant during the "approach to contact." Furthermore, we assume that additional X_i forces enter the "fields of fire" of the Y_i force at a rate denoted as $r_i(t)$. The above model might describe the combat attrition process for an amphibious assault (see [7], [8], [13]) in which the attacking side employs fire-support units.

The combat situation described above is diagrammatically shown in Figure 3. It is convenient to restate the problem as follows:

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T) \right\},$$

$$\text{with stopping rule: } t_f - T = 0,$$

$$\text{subject to: } \frac{dx_i}{dt} = -a_i y_i + r_i(t), \tag{42}$$

$$\frac{dy_i}{dt} = -\phi_i c_i y_i \quad \text{for} \quad i = 1,2,$$

---

†Alternatively, if small groups of defenders are attacked by the W fire-support units, then the same mathematical form of attrition occurs when the time to acquire targets is the constraining factor in the attrition process and this time is assumed to be inversely proportional to enemy troop density. Brackney [5] postulates that this occurs for attacks upon enemy defensive positions in which one must search (i.e. visually scan) for targets.

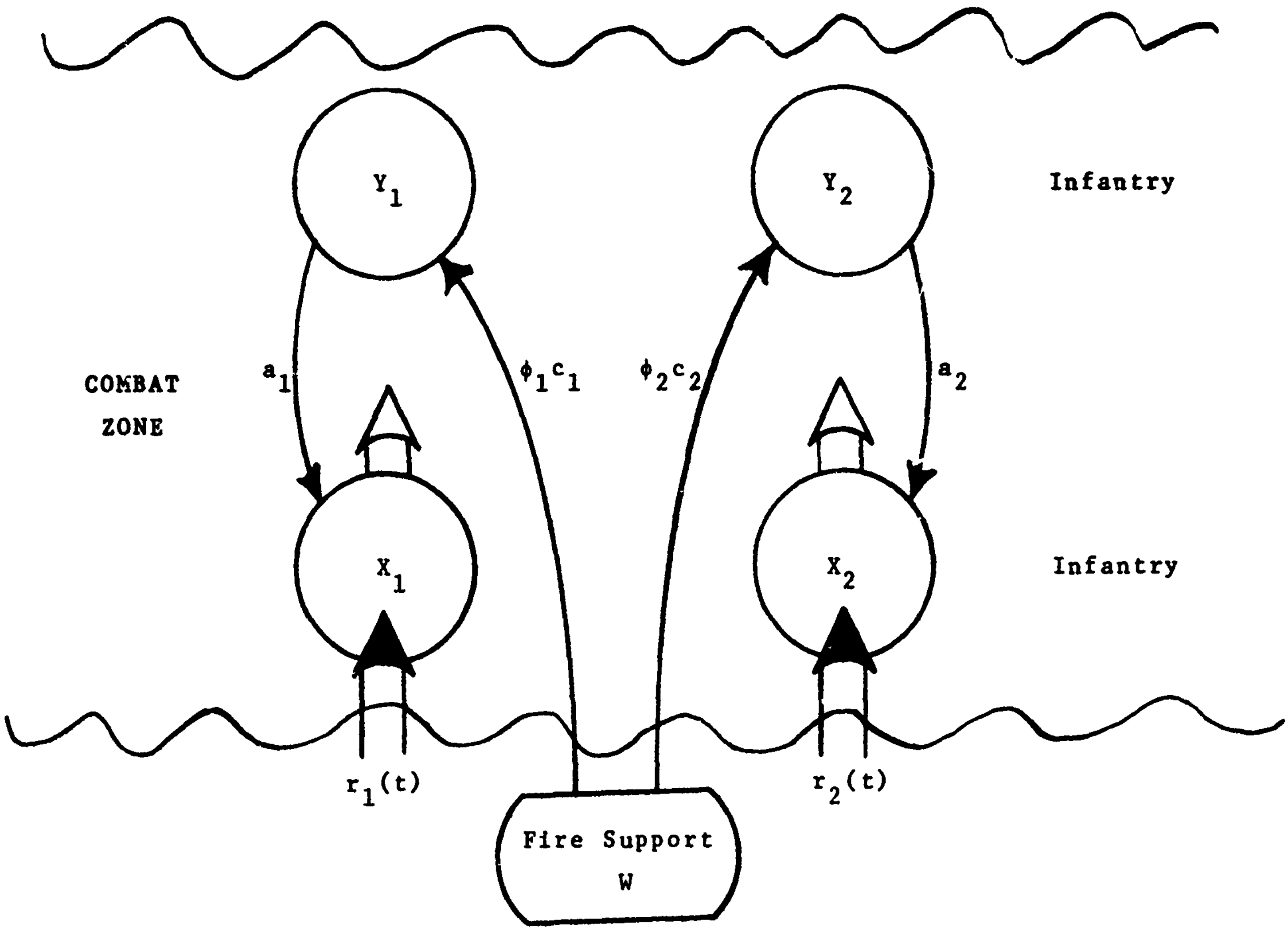


Figure 3. Diagram of Time-Sequential Fire-Support Problem for an Amphibious Assault (Denoted as Problem 2).

$$x_1, x_2, y_1, y_2 \geq 0 \quad \text{(State Variable Inequality Constraints)}$$

$$\phi_1 + \phi_2 = 1 \quad \text{and} \quad \phi_i \geq 0 \quad \text{for} \quad i = 1,2$$

$$\text{(Control Variable Inequality Constraints),}$$

where all symbols are as defined above. As shown in Table I, the above problem (42) has been designated as Problem 2. It will be again convenient to consider the single control variable ϕ defined by (3). It should be noted that for $T < +\infty$ it follows that we will always have $y_i(t) > 0$ for $i = 1,2$. Thus, the only state variable inequality constraints (SVIC's) that must be considered are $x_i \geq 0$. However, let us further assume that the attacker's infantry force levels are never reduced to zero. This might be militarily justified on the grounds that X would never attack the Y_i position if his attacking X_i forces could not survive the "approach to contact." As a possible future research task we would recommend the determination of what relationship between the Lanchester attrition-rate coefficients, initial force levels, length of approach to contact, and the X fire-support policy is sufficient to guarantee this (see Section 6.2.5). In Appendix C we consider some models in which breakpoints are considered for the various forces.

6.2.1. Optimal Fire-Support Policy.

The optimal time-sequential fire-support policy (expressed as a closed-loop control[†]) for Problem 2 is shown in Table II with ancillary information on switching times being given in Table III. It should be recalled that we have assumed that neither of the attacking infantry forces can be reduced to a zero force level during the approach to contact. The proofs of certain statements

[†]For a discussion of the distinction between an open-loop time-sequential policy and a closed-loop one, see [34] or [38]. For deterministic models such as the one under consideration, the two types of policies are well known to be equivalent.

Table II. Optimal Fire-Support Policy for Problem 2.[†]

Nonrestrictive Assumption: $w_1/(a_1v_1) \geq w_2/(a_2v_2)$

Optimal (closed-loop) time-sequential fire-support policy is

PHASE I for $0 \leq t < t_1 = T - \tau_1(y_1^f/y_2^f)$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for} \quad y_1/y_2 > a_2c_2v_2/(a_1c_1v_1), \\ c_2/(c_1+c_2) & \text{for} \quad y_1/y_2 = a_2c_2v_2/(a_1c_1v_1), \\ 0 & \text{for} \quad y_1/y_2 < a_2c_2v_2/(a_1c_1v_1), \end{cases}$$

PHASE II for $T - \tau_1(y_1^f/y_2^f) \leq t \leq T$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = 1,$$

where

$$\tau_1 = \begin{cases} \tau_S & \text{for} \quad \rho^f \geq \rho_S^f, \\ \tau_\phi & \text{for} \quad \rho_L \leq \rho^f < \rho_S^f, \\ 0 & \text{for} \quad \rho^f < \rho_L, \end{cases}$$

$$\rho = y_1/y_2, \qquad \text{and} \qquad \rho_L = \left(\frac{a_2c_2v_2}{a_1c_1v_1}\right)\left(\frac{w_2}{a_2v_2}\right) \Big/ \left(\frac{w_1}{a_1v_1}\right).$$

NOTES[††]:

(1) τ_S is the unique nonnegative root of $F(\tau=\tau_S) = 0$.

(2) For $\rho_L < \rho^f < \rho_S^f$, τ_ϕ is the smaller of the two positive roots of $G(\tau=\tau_\phi;\rho^f) = 0$.

[†]It is assumed that problem parameters and initial force levels are such that $x_i(T) > 0$ for $i = 1,2$.

[††]See Table III for the definitions of $F(\tau)$ and $G(\tau;\rho^f)$.

Table III. Determination of the Switching Times τ_S and τ_ϕ for Problem 2.

Nonrestrictive Assumption: $w_1/(a_1v_1) \geq w_2/(a_2v_2)$

τ_S is the unique nonnegative root of $F(\tau=\tau_S) = 0$.

For $\rho_L < \rho^f < \rho^f_S$, τ_ϕ is the smaller of the two positive roots of $G(\tau=\tau_\phi;\rho^f) = 0$.

It has been shown that

(a) bounds on τ_ϕ are given by $0 \leq \tau_\phi < \tau_S$,

(b) τ_ϕ is a strictly increasing function of ρ^f for $\rho_L \leq \rho^f < \rho^f_S$,

(c) there is no root to $G(\tau=\tau_\phi;\rho^f) = 0$ for $\rho^f > \rho^f_S$.

For the above we have

$$F(\tau) = \tau + \left(\frac{1}{c_1} - \frac{w_1}{a_1v_1}\right) e^{-c_1\tau} - \left(\frac{1}{c_1} - \frac{w_2}{a_2v_2}\right)$$

$$G(\tau;\rho^f) = \frac{1}{c_1}\left(e^{c_1\tau}-1\right)\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\rho^f - \tau + \left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\left(\frac{w_1}{a_1v_1}\right)\rho^f - \left(\frac{w_2}{a_2v_2}\right)$$

Bounds on τ_S are given by:

(a) For $w_1/(a_1v_1) \leq 1/c_1$,

$$\frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2} \leq \tau_S \leq \frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)\Big/\left(\frac{w_1}{a_1v_1}\right)\right\} .$$

(b) For $1/c_1 \leq w_1/(a_1v_1)$,

$$\frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)\Big/\left(\frac{w_1}{a_1v_1}\right)\right\} \leq \tau_S \leq \frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2} .$$

regarding switching times are to be found in Section 2.3 of Appendix B.

As a closed-loop control, the optimal fire-support policy is most conveniently expressed in terms of $y_1/y_2 = \rho$ (i.e., the ratio of the numerical strengths of the two defending infantry forces) and $\tau = T - t$ (i.e., the "backwards" time or "time to go" in the approach to contact). When enemy forces are valued in direct proportion to the rate at which they destroy value of the friendly forces, i.e.

$$w_i = ka_iv_i \quad \text{for} \quad i = 1,2, \tag{43}$$

the optimal fire-support policy takes a particular simple form (denoted as POLICY A):

POLICY A: For $0 \leq t \leq T$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for} \quad y_1/y_2 > a_2c_2v_2/(a_1c_1v_1), \\ c_2/(c_1+c_2) & \text{for} \quad y_1/y_2 = a_2c_2v_2/(a_1c_1v_1), \\ 0 & \text{for} \quad y_1/y_2 < a_2c_2v_2/(a_1c_1v_1). \end{cases} \tag{44}$$

This is shown pictorially in Figure 4 in which optimal trajectories are traced backwards in time. In this case, $\tau_1 = 0$ (see Table II), i.e. the entire approach to contact is "PHASE I." It is convenient to note that, for example, when $\phi(\tau)$ = CONSTANT for $0 \leq \tau \leq 0$, we have

$$\rho(\tau) = \rho^f \exp\{[\phi c_1 - (1-\phi)c_2]\tau\} .$$

When enemy forces are not valued in direct proportion to the rate of which they destroy value of the friendly forces (without loss of generality we may assume that $w_1/(a_1v_1 > w_2/(a_2v_2))$, the solution to Problem 2 is considerably more complex as shown in Figure 5. As we see from Table II, the planning horizon may be considered to consist of two phases (denoted as PHASE I and as PHASE II) during each of which a different fire-support allocation rule is

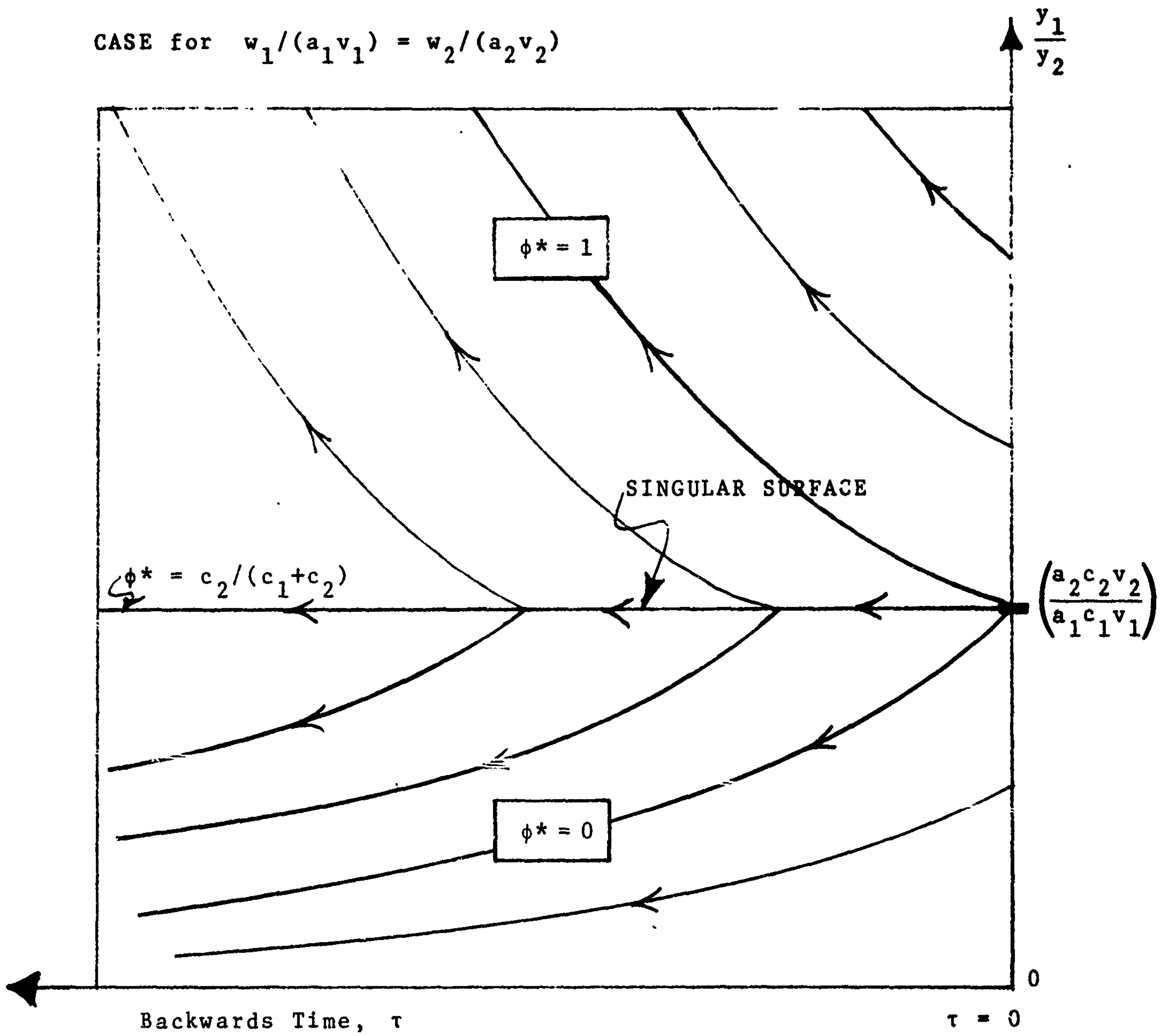
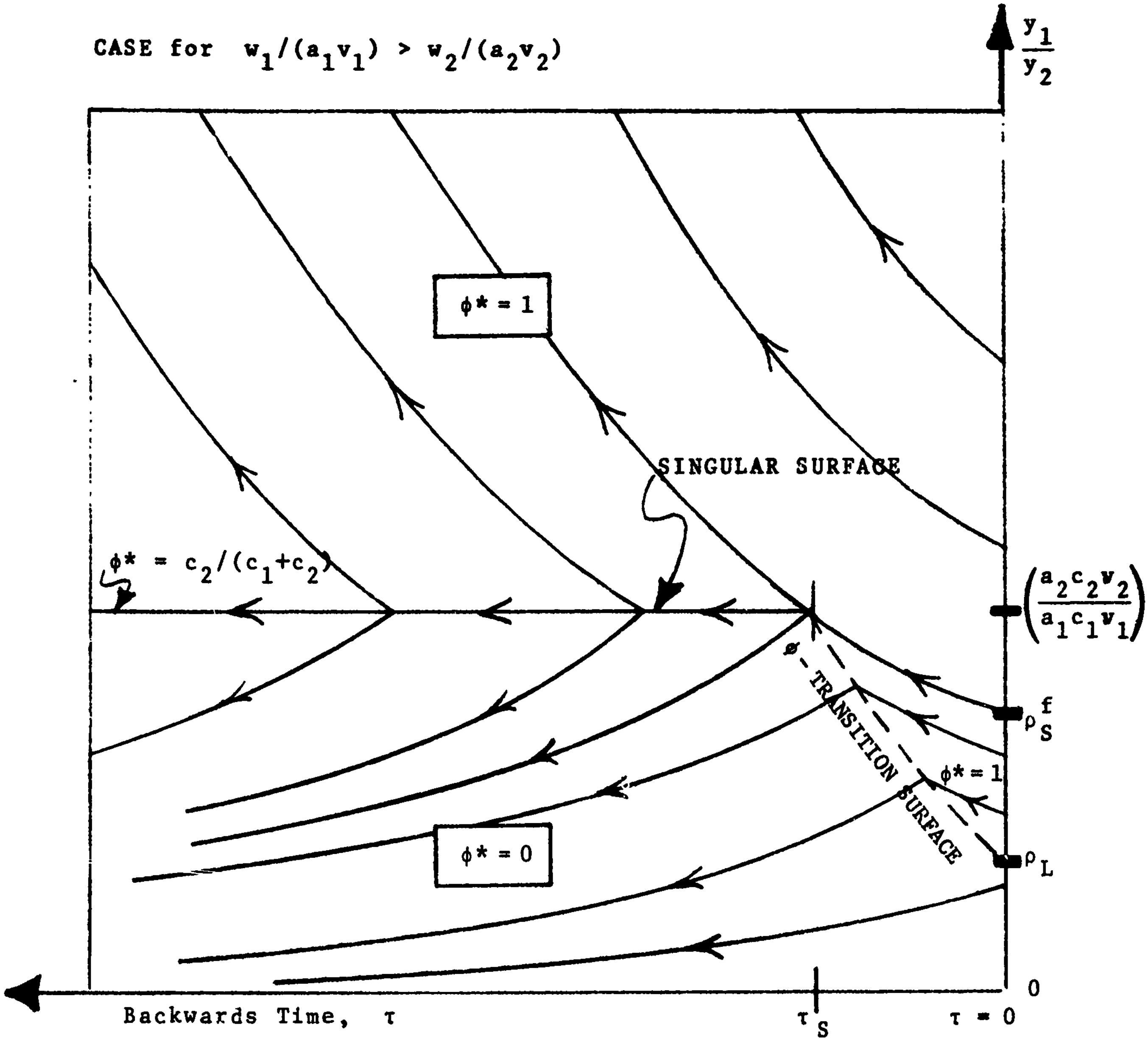


Figure 4. Diagram of Optimal (Closed-Loop) Fire-Support Policy (POLICY A) for Problem 2 When $w_1/(a_1v_1) = w_2/(a_2v_2)$ (not drawn to scale).



NOTES: (1) $\rho = y_1/y_2$,

(2) See Table III for definitions of ρ_L and ρ_S^f.

Figure 5. Diagram of Optimal (Closed-Loop) Fire-Support Policy (POLICY B) for Problem 2 When $w_1/(a_1v_1) > w_2/(a_2v_2)$ (not drawn to scale).

optimal. We denote this policy as POLICY B (see Table II). During PHASE I, POLICY A is optimal; while during PHASE II, it is optimal to concentrate all supporting fires on Y_1 (which has been valued disproportionately high). The absence or presence of PHASE II itself in the optimal time-sequential fire support policy depends on the ratio of enemy strengths $\rho = y_1/y_2$. The length of PHASE II (i.e. τ_1) is independent of the final force levels of the attacking infantry units (i.e. x_1^f and x_2^f) but depends only on $\rho^f = y_1^f/y_2^f$ and the combat effectiveness parameters (see equations (42)).

6.2.2. Necessary Conditions of Optimality.

The Hamiltonian [6] is given by (using (3))

$$H = \sum_{i=1}^{2} p_i(-a_i y_i + r_i(t)) - q_1 \phi c_1 y_1 - q_2(1-\phi)c_2 y_2, \tag{45}$$

so that the maximum principle again yields the extremal control law (5). The adjoint system of equations for the dual variable (again using (3) for convenience) is given by (assuming that $x_i > 0$)

$$p_i(t) = v_i \quad \text{for} \quad 0 \leq t \leq T,$$

and (46)

$$\dot{q}_i = a_i v_i + \phi_i^* c_i q_i \quad \text{with} \quad q_i(T) = -w_i \quad \text{for} \quad i = 1,2.$$

Computing the first two time derivatives of the switching function (6)

$$\dot{S}_\phi(t) = -a_1 c_1 v_1 y_1 + a_2 c_2 v_2 y_2 \ , \tag{47}$$

$$\ddot{S}_\phi(t) = a_1 c_1 v_1 y_1 (c_1 \phi) - a_2 c_2 v_2 y_2 (c_2(1-\phi)), \tag{48}$$

we see that on a singular subarc we have [6], [16]

$$y_1/y_2 = a_2c_2v_2/(a_1c_1v_1), \tag{49}$$

$$(-q_1)/(a_1v_1) = (-q_2)/(a_2v_2), \tag{50}$$

with the singular control given by

$$\phi_S = c_2/(c_1+c_2). \tag{51}$$

On such a singular subarc the gernalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1c_1v_1y_1(c_1+c_2) > 0$.

The adjoint variables $\underset{\sim}{p}(t)$ and $\underset{\sim}{q}(t)$ are continuous at all points of continuity of $\underset{\sim}{r}^T(t) = (r_1(t), r_2(t))$. Let t_d be a point of discontinuity of $\underset{\sim}{r}(t)$. Then again $\underset{\sim}{p}(t)$ and $\underset{\sim}{q}(t)$ are continuous at t_d, although the Hamiltonian satisfies $H(t_d^-) = H(t_d^+) - \nu$, where ν is an unrestricted multiplier. Thus, changes (discontinuous) in $\underset{\sim}{r}(t)$ have no direct effect on the optimal fire-support policy.

6.2.3. <u>Synthesis of Extremals</u>.

In synthesizing extremals by the usual backwards construction procedure it is convenient to consider (21) and

$$\dot{S}_\phi(\tau) = a_1c_1v_1y_1 - a_2c_2v_2y_2. \tag{52}$$

We will omit most of the tedious details of the synthesis of extremals because they are very similar to those given in [31]. Without loss of generality we may assume that $w_1/(a_1v_1) \geq w_2/(a_2v_2)$, and then there are two cases to be considered:

(I) $w_1/(a_1v_1) = w_2/(a_2v_2)$,

(II) $w_1/(a_1v_1) > w_2/(a_2v_2)$.

CASE I: $w_1/(a_1v_1) = w_2/(a_2v_2)$; i.e. $w_i = ka_iv_i$ for $i = 1,2$.

In this case (21) becomes

$$S_\phi(\tau=0) = a_2c_2v_2y_2^f(w_1/(a_1v_1))\left\{a_1c_1v_1y_1^f/(a_2c_2v_2y_2^f) - 1\right\},$$

whence follows (44) by the usual methods.

CASE II: $w_1/(a_1v_1) > w_2/(a_2v_2)$.

In this case it follows from (5), (6), and (21) that for $\rho^f = y_1^f/y_2^f \geq a_2c_2v_2/(a_1c_1v_1)$ we have $S_\phi(\tau) > 0$ and $\phi^*(\tau) = 1$ for all $\tau > 0$. Since $S_\phi(\tau=0) \leq 0 \Longrightarrow \mathring{S}_\phi(\tau=0) < 0$, it follows that for $\rho^f \leq \left(\frac{a_2c_2v_2}{a_1c_1v_1}\right)\left(\frac{w_2}{a_2v_2}\right)\Big/\left(\frac{w_1}{a_1v_1}\right)$ we have $S_\phi(\tau) < 0$ and $\phi^*(\tau) = 0$ for all $\tau > 0$.

There may be a change in the sign of $S_\phi(\tau)$ for $c_2w_2/(c_1w_1) < \rho^f < a_2c_2v_2/(a_1c_1v_1)$. In this case $\phi^*(\tau) = 1$ for $0 \leq \tau \leq \tau_1$ and then

$$S_\phi(\tau) = a_2c_2v_2y_2^f\left\{\frac{1}{c_1}\left(e^{c_1\tau}-1\right)\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\rho^f - \tau + \left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\left(\frac{w_1}{a_1v_1}\right)\rho^f - \left(\frac{w_2}{a_2v_2}\right)\right\}.$$

It is clear that we must have $\mathring{S}_\phi(\tau=\tau_1) \leq 0$. If $\mathring{S}_\phi(\tau=\tau_1) < 0$, then we have a transition surface with τ_1 (denoted as τ_ϕ) given by the smaller of the two positive roots of $G(\tau=\tau_\phi;\rho^f) = 0$, where $G(\tau;\rho^f)$ is given in Table III. If $\mathring{S}_\phi(\tau=\tau_1) = 0$, the singular subarc may be entered, and then τ_1 (denoted as τ_S) is given by the unique nonnegative root of $F(\tau=\tau_S) = 0$, where $F(\tau)$ is given in Table III. We denote the corresponding value of ρ^f as ρ_S^f. Then there is no switch in ϕ^* for $\rho^f > \rho_S^f$ (see Section 4.2 of Appendix B for a proof of this statement; the development of bounds for τ_S is also given there).

The above information immediately leads to the extremal field shown in Figure 5 (see also Tables II and III).

6.2.4. Determination of the Optimal Fire-Support Policy.

As we have discussed elsewhere [30]-[32], [34], [38], the optimality of an extremal trajectory may be proven by citing the appropriate existence theorem for an optimal control to the problem at hand; there are two further subcases: (1) if the extremal is unique, then it is optimal or (2) if the extremal is not unique and only a finite number exist, then the optimal trajectory is determined by considering the finite number of corresponding values of the criterion functional.† The existence of a measurable optimal control follows by Corollary 2 on p. 262 of [18]. In Section 6.2.2 and 6.2.3 above, we have considered necessary conditions of optimality for piecewise continuous admissible controls (see p. 10 and pp. 20-21 of [24]). It remains to show that one of the measurable optimal controls is piecewise continuous. This may be done by observing that if we consider the maximum principle for measurable controls†† (see p. 81 of [24]), then it follows from the backwards synthesis of extremals that the optimal control is piecewise constant (and hence piecewise continuous)†††. The optimality of the extremal fire-support policy developed above follows by the uniqueness of extremals (see [31]).

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†It has not been possible to determine the optimality of a policy by citing one of the many known sets of sufficient conditions (see [6], [31], [38]). In particular, even though the planning horizon for the problem at hand is of fixed length, one cannot invoke the sufficient conditions based on convexity of Mangasarian [20] or Funk and Gilbert [9] because the right-hand sides of the differential equations (42) are not concave functions of x_i, y_i, and ϕ_i.

††We have taken the liberty of changing the sign of the adjoint vector of Pontryagin et al. [24] (see p. 108 of [6]). When the admissible controls are measurable and bounded, the Hamiltonian (45) need only attain its maximum almost everywhere in time.

†††This follows from the control variable appearing linearly in the Hamiltonian (45), the control variable space being compact, and the switching function being continuous for $0 \leq t \leq T$. The maximum principle (also singular control considerations) then yields that the optimal control must be piecewise constant and uniquely determined almost everywhere, since $S_\phi(t)$ can change sign at most once (see p. 130 of [24]). [The author wishes to thank J. Wingate for pointing out this type of argument.]

6.2.5. A Further Consequence of Fire Support.

In the work at hand we have examined optimal fire-support policies under the assumption that $x_i > 0$. Another aspect that we will briefly discuss here (but not at this time pursue further) is the quantification of how fire support can guarantee that $x_i > 0$ always. From the state equations (42) we have

$$x_i(t) = x_i^o + \int_0^t r_i(s)ds - a_i y_i^o \int_0^t ds_1 \exp\{-c_i \int_0^{s_1} \phi_i(s_2)ds_2\} ,$$

so that we see explicitly how $x_i(t)$ depends on the fire-support policy adopted by X . For example, when $\phi_i = 1$ and $r_i(t) = \bar{r}_i$, then

$$x_i(t) = x_i + \bar{r}_i\, t - (1-e^{-c_i t})a_i y_i^o/c_i.$$

For such an expression, it would be of interest to determine what conditions guarantee that $x_i(t) \geq K \geq 0$. Moreover, time-sequential fire-support allocation in this model may determine whether enough X_i survive the approach to contact to effectively initiate close-assault tactics.

6.3. Problem 2a.

As seen in Table I, Problem 2a is the version of Problem 2 in which $r_i(t) \equiv 0$. This problem is further considered in Appendix B within the context of examining the influence of X's combat objectives on his time-sequential fire-support policy. The optimal fire-support policy for Problem 2a (under the assumption that $x_i > 0$) is exactly the same as that for Problem 2 (as given in Section 6.2.1 above).

6.4 Problem 2b.

It is of interest to consider a version of Problem 2a with temporal variations in the effectiveness of Y_i's fire. This might model, for example, the situation in which the X_i forces move as a fairly compact unit and the effectiveness of Y_i's fire is strongly dependent upon the force separation between X_i and Y_i. [We recall that the X_i forces are moving towards the static position of Y_i (see Figure 3).] Let us briefly consider this case.

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k (T) - \sum_{k=1}^{2} w_k y_k (T) \right\},$$

with stopping rule: $t_f - T = 0$.

$$\text{subject to: } \frac{dx_i}{dt} = -a_i(t)y_i ,$$

$$\frac{dy_i}{dt} = -\phi_i c_i y_i \quad \text{for} \quad i = 1,2, \tag{53}$$

$$x_i,\ y_i \geq 0 , \quad \phi_1 + \phi_2 = 1 , \quad \text{and} \quad \phi_i \geq 0 \quad \text{for} \quad i = 1,2.$$

It will again sometimes be convenient to consider the single control variable ϕ defined by (3). As usual, we consider only the case in which $x_1(T) > 0$.

6.4.1. Optimal Fire-Support Policy in a Special Case.

When enemy survivors are valued in direct proportion to the rate at which they destroy value of the friendly forces (i.e. (43) holds), the optimal fire-support policy takes a particularly simple form: for $0 \leq t \leq T$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for} \quad y_1/y_2 > a_2(t)c_2v_2/(a_1(t)c_1v_1), \\ \phi_S & \text{for} \quad y_1/y_2 = a_2(t)c_2v_2/(a_1(t)c_1v_1), \\ 0 & \text{for} \quad y_1/y_2 < a_2(t)c_2v_2/(a_1(t)c_1v_1). \end{cases} \tag{54}$$

where

$$\phi_S = c_2/(c_1+c_2) + \left\{\frac{1}{a_1(t)} \frac{da_1}{dt} - \frac{1}{a_2(t)} \frac{da_2}{dt}\right\} \Big/ (c_1+c_2). \tag{55}$$

It should be noted that when $a_1(t)/a_2(t)$ = constant, then $\phi_S = c_2/(c_1+c_2)$ so that the solution is essentially the same as that for Problem 2 in this case.

6.4.2. Necessary Conditions of Optimality.

The necessary conditions of optimality for (53) are similar to those for Problem 2 given in Section 6.2.2 above. The Hamiltonian is given by

$$H = -\sum_{i=1}^{2} p_i a_i(t) y_i - q_1 \phi c_1 y_1 - q_2(1-\phi) c_2 y_2. \tag{56}$$

The maximum principle again yields (5) as the extremal control law, and (46) again holds (with a_i replaced by $a_i(t)$).

Computing the first two time derivatives of the switching function (6)

$$\dot{S}_\phi(t) = -a_1(t) c_1 v_1 y_1 + a_2(t) c_2 v_2 y_2,$$

$$\ddot{S}(t) = a_1(t) c_1 v_1 y_1 (c_1 \phi) - a_2(t) c_2 v_2 y_2 (c_2(1-\phi)) - \dot{a}_1 c_1 v_1 y_1 + \dot{a}_2 c_2 v_2 y_2 ,$$

we see that on a singular subarc [6], [16]

$$y_1/y_2 = a_2(t) c_2 v_2/(a_1(t) c_1 v_1),$$

$$(-q_1)/(a_1(t) v_1) = (-q_2)/(a_2(t) v_2),$$

with the singular control given by (55). The generalized Legendre-Clebsch condition is easily shown to hold.

The results given in Section 6.4.1 follow from the usual backwards synthesis procedure and the observation that (21) still holds.

6.4.3. Suggested Future Work.

It would be of interest to examine the optimal time-sequential fire-support policy when $w_1/(a_1^f v_1) > w_2/(a_2^f v_2)$ (where $a_1(T) = a_1^f$). We suggest this as a possible future research task.

Let us consider some of the algebraic complexities of the above proposed work. We will focus on the determination of the switching time τ_1 corresponding to that given in Section 6.2.3 above. In this case $\phi^*(\tau) = 1$ for $0 \le \tau \le \tau_1$, and then

$$S_\phi(\tau) = a_2^f c_2 v_2 y_2^f \left\{ \left(\frac{a_1^f c_1 v_1}{a_2^f c_2 v_2}\right)\left(\frac{w_1}{a_1^f v_1}\right)\rho^f - \left(\frac{w_2}{a_2^f v_2}\right) + \left(\frac{a_1^f c_1 v_1}{a_2^f c_2 v_2}\right)\rho^f \int_0^\tau e^{c_1\sigma} \cdot \{a_1(\sigma)/a_1^f\} d\sigma - \int_0^\tau \{a_2(\sigma)/a_2^f\} d\sigma \right\},$$

where $\rho^f = y_1^f/y_2^f$.

One must make assumptions about the functional form of $a_i(t)$ to carry the above work along further. If, for example, $a_i(t) = k_{a_i} t^{m_i}$, where m_i is a positive integer, then $a_i(\sigma) = k_{a_i}(T-\sigma)^{m_i}$. We then have

$$\int_0^\tau e^{c_1\sigma} \cdot \{a_1(\sigma)/a_1^f\} d\sigma =$$

$$\frac{1}{c_1} \sum_{k=0}^{m_1} \frac{m_1!}{(m_1-k)!} \frac{1}{(c_1 T)^k} \{(1-\tau/T)^{m_1-k} e^{c_1\tau} - 1\} .$$

It appears to be very messy to determine τ_1 such that $S_\phi(\tau=\tau_1) = 0$, but numerical methods might prove useful here.

6.5. Problem 3.

In this section we will consider a version[†] (see Section 6.1.4) of Problem 1 as given by (2) in which the Y forces attack the static defense of the X forces along a "front." We assume that the X_i force causes attrition to the Y_i force according to a "square-law" attrition process.[††] The attrition of the X_i force by the attacking Y_i is assumed to be negligible. As before, the X fire-support units (denoted as W) deliver "area fire" against the Y_i forces. All Lanchester attrition-rate coefficients are assumed to be constant during the "approach to contact." Furthermore, we assume that additional Y_i forces enter the "fields of fire" of the X_i forces at a constant rate denoted as s_i.

The combat situation described above is diagrammatically shown in Figure 6. It is convenient to restate the problem as follows:

$$\underset{\phi_i(t)}{\text{minimize}} \sum_{k=1}^{2} w_k y_k(T),$$

with stopping rule: $t_f - T = 0$,

$$\text{subject to: } \frac{dy_i}{dt} = K_i - \phi_i c_i y_i \tag{57}$$

$y_i \geq 0$, $\phi_1 + \phi_2 = 1$, and $\phi_i \geq 0$ for $i = 1,2$,

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[†]This is, of course, a special case of the general problem (1) (see Section 4) graphically depicted in Figure 1 in which the following hold:

$b_i(t,y_i) = \overline{b}_i$ = constant,
$B_i(t,y_i) = c_i y_i$ where c_i is constant,
$s_i(t) = \overline{s}_i$ = constant,
and $a_i(t,x_i) = A_i(t,x_i) = B(t,z) = r_i(t) = 0$.

For notational convenience we will again denote $\overline{b}_i$ as b_i, etc.

[††]See Section 6.2 for a discussion of the rationale for this and subsequent assumptions.

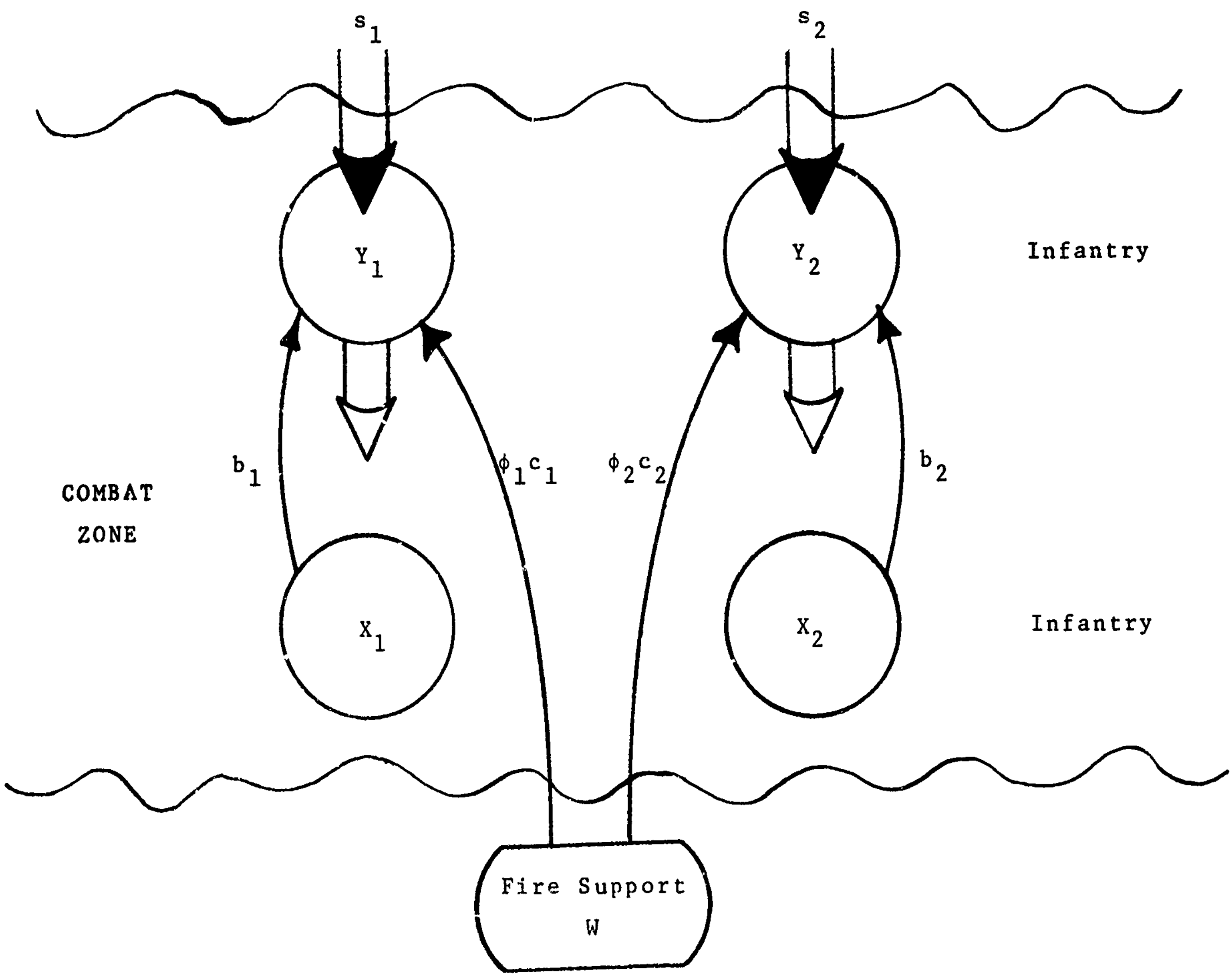

Figure 6. Diagram of Time-Sequential Fire-Support Problem For an Enemy Attack (Denoted as Problem 3).

where

$$K_i = -b_i x_i^o + s_i \ . \tag{58}$$

As above, it will sometimes be convenient to consider the single control variable ϕ defined by (3). Again, we consider only the case in which $y_i > 0$.

6.5.1. Necessary Conditions of Optimality.

The Hamiltonian is given by

$$H = q_1(K_1-\phi c_1 y_1) + q_2(K_2-(1-\phi)c_2 y_2). \tag{59}$$

The maximum principle again yields (5) as the extremal control law, and the adjoint equations are (for $y_i > 0$)

$$\dot{q}_i = \phi_i^* c_i q_i \quad \text{with} \quad q_i(T) = w_i \quad \text{for} \quad i = 1,2. \tag{60}$$

Computing the first two time derivatives of the switching function (6)

$$\dot{S}_\phi(t) = c_1 K_1 q_1 - c_2 k_2 q_2 \ ,$$

$$\ddot{S}_\phi(t) = c_1 K_1 q_1 (c_1 \phi) - c_2 K_2 q_2 (c_2(1-\phi)),$$

we see that on a singular subarc

$$y_1/y_2 = K_1/K_2 = (s_1-b_1 x_1^o)/(s_2-b_2 x_2^o), \tag{61}$$

$$c_1 K_1 q_1 = c_2 K_2 q_2, \tag{62}$$

with the singular control again given by (51): $\phi_S = c_2/(c_1+c_2)$. On such a singular subarc we have $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = -c_1 K_1 q_1 (c_1+c_2)$ so that the generalized Legendre-Clebsch condition is satisfied only when $K_1, K_2 \geq 0$, since $q_i(t) > 0$ $\forall t$ for $i = 1,2$.

In synthesizing extremals it is convenient to consider

$$S_\phi(\tau=o) = c_2K_2w_2\left\{\left(\frac{c_1K_1w_1}{c_2K_2w_2}\right)\frac{y_1^f}{K_1} - \frac{y_2^f}{K_2}\right\} , \tag{63}$$

and

$$\overset{o}{S}_\phi(\tau) = -c_1K_1q_1 + c_2K_2q_2 . \tag{64}$$

From consideration of the generalized Legendre-Clebsch condition we see that three cases to be considered (others are possible) are:

(A) $K_1, K_2 > 0$,

(B) $K_1 = K_2 = 0$,

(C) $K_1, K_2 < 0$.

Singular subarcs are not optimal for Case C.

6.5.2. Optimal Fire-Support Policy for $b_ix_i^o < s_i$ for $i = 1,2$.

Without loss of generality we may assume that $c_1K_1w_1 \geq c_2K_2w_2$, and then there are two cases to be considered:

(I) $c_1K_1w_1 = c_2K_2w_2$,

(II) $c_1K_1w_1 > c_2K_2w_2$.

CASE I: $c_1K_1w_1 = c_2K_2w_2$; i.e. $w_i = k/(c_iK_i)$ for $i = 1,2$.

In this case enemy survivors are valued inversely proportional to the product of their vulnerability to W's fire and net rate of change exclusive of W fire support. Then (63) becomes $S_\phi(\tau=0) = c_2K_2w_2\{y_1^f/K_1 - y_2^f/K_2\}$. Noting that $\frac{d}{d\tau}\left(\frac{y_1}{K_1} - \frac{y_2}{K_2}\right) = \phi c_1\left(\frac{y_1}{K_1}\right) - (1-\phi)c_2\left(\frac{y_2}{K_2}\right)$, the usual arguments yield that the optimal time-sequential fire-support policy is given by: for $0 \leq t \leq T$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for } y_1/y_2 > K_1/K_2 , \\ c_2/(c_1+c_2) & \text{for } y_1/y_2 = K_1/K_2 , \\ 0 & \text{for } y_1/y_2 < K_1/K_2 . \end{cases}$$

CASE II: $c_1K_1w_1 > c_2K_2w_2$.

In this case the solution has the same structure as that for Problem 2 as given in Tables II and III. The planning horizon may be considered to be divided into two phases in a similar fashion. Details are to be worked out in the future.

6.5.3. Optimal Fire-Support Policy for $b_i x_i^o = s_i$ for $i = 1,2$.

In this case $K_1 = K_2 = 0$ so that (6), (60) and (64) yield that

$$S_\phi(t) = c_1 w_1 y_1^f - c_2 w_2 y_2^f ,$$

whence follows that the optimal fire-support policy is given by: for $0 \le t \le T$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for } \rho \ge \rho_S^f e^{c_1(T-t)} , \\ \text{any feasible value} & \text{for } \rho_S^f e^{-c_2(T-t)} < \rho < \rho_S^f e^{c_1(T-t)} , \\ 0 & \text{for } \rho \le \rho_S^f e^{-c_2(T-t)} , \end{cases} \qquad (65)$$

where $\rho = y_1/y_2$, $\rho_S^f = \left(\frac{a_2c_2v_2}{a_1c_1v_1}\right)\left(\frac{w_2}{a_2v_2}\right)\Big/\left(\frac{w_1}{a_1v_1}\right) = c_2w_2/(c_1w_1)$, and $\dot{\rho} = \{-\phi c_1 + (1-\phi)c_2\}\rho$. An understanding of this case is essential for developing the solution for the next case.

6.5.4. Optimal Fire-Support Policy for $b_i x_i^o > s_i$ for $i = 1,2$.

In this case it is never optimal for the W fire-support units to split their fire between Y_1 and Y_2 (for a finite interval of time). The development of a solution is much more complex than that for the previous cases and depends on an understanding of the results given in Section 6.5.3. Details have not been completely worked out at the time of the writing of this report, and we would propose to ONR as possible future research the further study of this important problem.

6.6. Problem 4.

As seen in Table I, Problem 4 is a version of Problem 1 (see Figure 2) in which we may consider the X force to be the attacker and the Y force to be the defender. Additionally, the attacking X_i force causes attrition to the defending Y_i according to a "linear-law" process[†], while the defending Y_i force causes attrition to the attacking X_i according to a "square-law" process.[††] Other aspects are the same as those for Problem 1. Thus, we have

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T) \right\},$$

$$\text{with stopping rule: } t_f - T = 0 ,$$

$$\begin{aligned} \text{subject to: } \frac{dx_i}{dt} &= -a_i y_i, \\ \frac{dy_i}{dt} &= -b_i y_i x_i - \phi_i c_i y_i , \end{aligned} \tag{66}$$

$$x_i,\ y_i \geq 0 \ ,\ \phi_1 + \phi_2 = 1 \ ,\ \text{ and }\ \phi_i \geq 0 \ \text{ for } \ i = 1,2 \ .$$

As above, it will sometimes be convenient to consider the single control variable ϕ defined by (3). For $T < +\infty$, it follows that $y_i(T) > 0$ so that we need

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[†]See Section 6.1 for an explanation of terminology.

[††]Brackney has hypothesized that such a situation occurs when both sides use aimed fire, a defender's time to acquire an attacker is negligible in comparison to the time to kill an acquired target, and the time for an attacker to acquire a defender is relatively large by reason of his opponent's remaining under cover in defensive positions (see pp. 32-33 of [5]).

only be concerned with the SVIC's $x_i \geq 0$ for $i = 1,2$. Again, we will consider only the case in which $x_i(T) > 0$.

6.6.1. Optimal Fire-Support Policy in a Special Case.

When enemy (i.e. Y) survivors are valued in direct proportion to the rate at which they destroy value of the friendly forces (i.e. (43) holds) <u>and</u> friendly survivors are valued in direct proportion to the ratio of their fire effectiveness to that of their supporting weapons (i.e. we have that

$$v_i = K\, b_i/c_i \quad \text{for} \quad i = 1,2), \tag{67}$$

the optimal fire-support policy takes a particularly simple form (assuming that $-c_1 < b_1x_1 - b_2x_2 < c_2)^{\dagger}$: for $0 \leq t \leq T$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for} \quad \rho > \rho_S^f , \\ \phi_S & \text{for} \quad \rho = \rho_S^f , \\ 0 & \text{for} \quad \rho < \rho_S^f , \end{cases} \tag{68}$$

where $\rho = y_1/y_2$, $\rho_S^f = a_2c_2v_2/(a_1c_1v_1)$, and $\dot{\rho} = \{-b_1x_1 + b_2x_2 - \phi c_1 + (1-\phi)c_2\}\rho$. When $b_1 = b_2 = 0$, we see that the solution reduces to that for Problem 2a.

6.6.2. Necessary Conditions of Optimality.

The Hamiltonian is given by

$$H = -\sum_{i=1}^{2} p_i a_i y_i - q_1y_1(b_1x_1+\phi c_1) - q_2y_2(b_2x_2 + (1-\phi)c_2). \tag{69}$$

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†This condition is satisfied if, for example, $|b_1x_1 - b_2x_2| <$ minimum (c_1,c_2).

The maximum principle again yields (5) as the extremal control law, and the adjoint equations are (for $x_i(T) > 0$)

$$\dot{p}_i = b_i y_i q_i \qquad \text{with} \quad p_i(T) = v_i$$

and (70)

$$\dot{q}_i = a_i p_i + (b_i x_i + \phi_i^* c_i) q_i \quad \text{with} \quad q_i(T) = -w_i \quad \text{for} \quad i = 1,2.$$

Computing the first two time derivatives of the switching function (6)

$$\dot{S}_\phi(t) = -a_1 c_1 p_1 y_1 + a_2 c_2 p_2 y_2 \ ,$$

$$\ddot{S}_\phi(t) = c_1(-q_1) y_1 (a_1 b_1 y_1) + a_1 c_1 p_1 y_1 (b_1 x_1 + \phi c_1)$$

$$-c_2(-q_2) y_2 (a_2 b_2 y_2) - a_2 c_2 p_2 y_2 (b_2 x_2 + (1-\phi) c_2) \ ,$$

we see that on a singular subarc

$$y_1/y_2 = a_2 c_2 p_2/(a_1 c_1 p_1) \ , \tag{71}$$

$$(-q_1)/(a_1 p_1) = (-q_2)/(a_2 p_2) \ , \tag{72}$$

with the singular control given by

$$\phi_S = c_2/(c_1+c_2) + \{b_2 x_2 - b_1 x_1 + (a_2 b_2 y_2 - a_1 b_1 y_1)(-q_1)/(a_1 p_1)\}/(c_1+c_2) \ . \tag{73}$$

The generalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1 c_1 p_1 y_1 (c_1+c_2) > 0$.

In synthesizing extremals it is convenient to consider (21),

$$\overset{\circ}{S}_\phi(\tau) = a_1 c_1 p_1 y_1 - a_2 c_2 p_2 y_2 \ ,$$

and

$$\overset{\circ\circ}{S}_{\phi}(\tau=0) = a_1c_1v_1y_1^f\left\{a_1c_1v_1y_1^f\left(\frac{w_1}{a_1v_1}\right)\left(\frac{b_1}{c_1v_1}\right) + b_1x_1 + \phi c_1\right\}$$

$$- a_2c_2v_2y_2^f\left\{a_2c_2v_2y_2^f\left(\frac{w_2}{a_2v_2}\right)\left(\frac{b_2}{c_2v_2}\right) + b_2x_2 + (1-\phi)c_2\right\},$$

whence follows the results given in Section 6.6.1 by the usual arguments using (43) and (67).

6.7. Problem 5.

As seen from Table I and Figure 1, Problem 5 is a version of the general fire-support problem (1) which corresponds to the addition of fire-support units (denoted as Z) to the enemy Y forces in the basic scenario of Problem 1 (see (2) and Figure 2). These Z fire-support units engage the X forces and cause attrition to X_i according to a "square-law" attrition process† with the corresponding Lanchester attrition-rate coefficient being denoted as α_i. Additionally, when the W fire-support units engage the enemy Z units in counter-battery fire, we assume that an enemy fire-support unit is engaged as a point target and that the W units have the capability to sense when an enemy supporting unit has been destroyed so that fire may be immediately shifted to a new target (with the W fire uniformly distributed over the Z survivors). Thus, we have the following fire-support problem

†This corresponds to assuming that "small groups" of the X_i force are attacked as point targets by the Z fire-support units and that the time to acquire such targets is negligible compared with the time to destroy them. Brackney [5] postulates that such is the case when the X_i forces assult the Y_i positions.

$$\underset{\phi_i(t)}{\text{maximize}}\left\{\sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)\right\} ,$$

with stopping rule: $t_f - T = 0$,

subject to:
$$\frac{dx_i}{dt} = -a_i y_i - \alpha_i z ,$$

$$\frac{dy_i}{dt} = -b_i x_i - \phi_i c_i y_i , \tag{74}$$

$$\frac{dz}{dt} = -(1-\phi_1-\phi_2)\beta ,$$

$$x_i, y_i, z \geq 0, \quad \phi_1+\phi_2 \leq 1, \quad \text{and} \quad \phi_i \geq 0 \quad \text{for} \quad i = 1,2 .$$

For the analysis presented here, we assume that $x_i(T)$, $y_i(T)$, and $z(T) > 0$.

As was the case for Problem 1 (see Sections 6.1.1 and 6.1.2 above), without the use of simplifying approximations (see Section 6.1.4) the determination of the optimal fire-support policy is (hopelessly) complex. Let us note here, however, that the Hamiltonian is given by (for $x_i, y_i, z > 0$)

$$H = -\sum_{i=1}^{2} p_i(a_i y_i + \alpha_i z) - \sum_{i=1}^{2} q_i(b_i x_i + \phi_i c_i y_i) - p(1-\phi_1-\phi_2)\beta .$$

The maximum principle yields that

$$\phi_1^*(t) = \begin{cases} 1 & \text{for } K_1 > \max(0,K_2) , \\ 0 & \text{for } K_1 < \max(0,K_2) , \end{cases} \tag{75}$$

and

$$\phi_2^*(t) = \begin{cases} 1 & \text{for } K_2 > \max(0,K_1) , \\ 0 & \text{for } K_2 < \max(0,K_1) , \end{cases} \tag{76}$$

where

$$K_i = c_i(-q_i)y_i - \beta(-p). \tag{77}$$

The adjoint equation are given by (7) and (assuming that $z(T) > 0$)

$$\dot{p} = \sum_{k=1}^{2} \alpha_k p_k \qquad \text{with} \qquad p(T) = 0 . \tag{78}$$

In the next section we consider the simpler case of the above problem (74) in which $b_i = 0$.

6.8. Problem 6.

As seen from Table I, Problem 6 is a simplified version of Problem 5 (setting $b_i = 0$ and adding friendly replacements at a rate denoted as $r_i(t)$). Problem 6 is analogous to Problem 2 with the addition of enemy fire support units (denoted as Z)(see Section 6.7 above). Thus, we have

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T) \right\} ,$$

$$\text{with stopping rule:} \quad t_f - T = 0 ,$$

$$\begin{aligned} \text{subject to:} \quad \frac{dx_i}{dt} &= -a_i y_i - \alpha_i z + r_i(t) , \\ \frac{dy_i}{dt} &= -\phi_i c_i y_i , \\ \frac{dz}{dt} &= -(1-\phi_1-\phi_2)\beta , \end{aligned} \tag{79}$$

$$x_i, y_i, z \geq 0, \qquad \phi_1 + \phi_2 \leq 1, \qquad \text{and} \qquad \phi_i \geq 0 \qquad \text{for} \qquad i = 1,2 .$$

For $T < +\infty$, we have $y_i(t) > 0$ always so that the only SVIC's that must be considered are x_i, $z \geq 0$ for $i = 1,2$. In the analysis presented here, however, we will assume that x_i , $z > 0$.

The optimal time-sequential fire-support policy[†] in the special case in which $z^f = z(T) > 0$ and enemy survivors are valued in direct proportion to the rate at which they destroy value of the friendly forces (i.e. (43) holds) is shown in Figure 7. It is impossible for it to be optimal for W to divide his fire between X_i and Z for a finite interval of time, since $\ddot{K}_i(t) = a_i c_i v_i y_i (c_i \phi_i) \neq 0$. However, it is possible to have an $X_1 - X_2$ split. Considering $D(t) = K_1(t) - K_2(t)$, we find that (49) and (50) again hold on such a singular subarc with the singular control given by (from $\ddot{D}(t) = 0$ when $\dot{D}(t) = 0$) $\phi_1^S = c_2/(c_1 + c_2)$ and $\phi_2^S = c_1/(c_1 + c_2)$.

In the future, we will give information for the various extremals shown in Figure 7 (see Figure 4 and Table I on pp. D-43 through D-55 of [37]). For example, for path P_{A1}^{CI} we have $y_1^f/y_2^f > a_2c_2v_2/(a_1c_1v_1)$ and[††]

$$\begin{cases} \phi_1^*(\tau) = 1 , \\ \phi_2^*(\tau) = 0 . \end{cases} \qquad \text{for} \quad 0 \leq \tau < \tau_1^{A1} ,$$

The switching time τ_1^{A1} is given by $K_1(\tau = \tau_1^{A1}) = 0$ where

$$K_1(\tau) = a_1 v_1 y_1^f \,(e^{c_1 \tau} - 1) - \beta \left(\sum_{k=1}^{2} \alpha_k v_k \right) \tau + c_1 w_1 y_1^f \quad .$$

It may be shown that $K_1(\tau) > K_2(\tau)$ for $0 \leq \tau \leq \tau_1^{A1}$.

[†]The details of the development of the optimal time-sequential fire-support policies for Problems 6 through 10 are omitted. These will be given in the future.

[††]The maximum principle again yields (75) and (76).

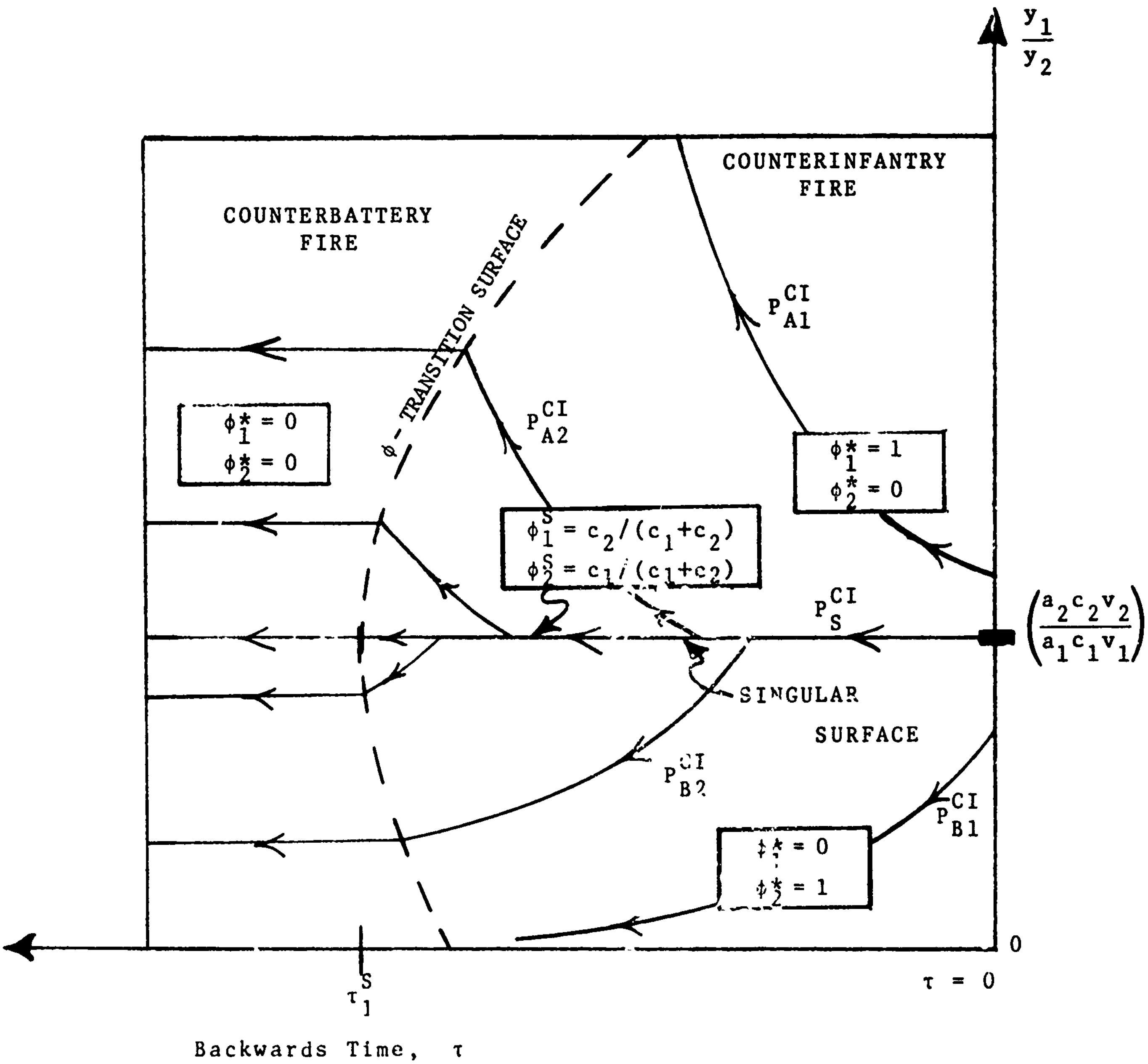

Figure 7. Diagram of Optimal (Closed-Loop) Fire-Support Policy for Problem 6 when $z^f > 0$ and $w_i = ka_i v_i$ for $i = 1,2$ (not drawn to scale).

6.9. Problem 7.

If we let the attrition by the enemy Z fire-support units in Problem 6 be a "linear-law" attrition process (see Sections 6.1 and 6.2 for a discussion of this assumption), then the resulting problem we denote as Problems 7 (see Table I). We have then

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T) \right\} ,$$

with stopping rule: $t_f - T = 0$,

$$\text{subject to: } \frac{dx_i}{dt} = -a_i y_i - \alpha_i x_i z ,$$

$$\frac{dy_i}{dt} = -\phi_i c_i y_i , \tag{80}$$

$$\frac{dz}{dt} = -(1 - \phi_1 - \phi_2)\beta ,$$

$$x_i, y_i, z \geq 0 , \quad \phi_1 + \phi_2 \leq 1 , \text{ and } \phi_i \geq 0 \quad \text{for} \quad i = 1,2 .$$

For $T < +\infty$, we have $y_i(T) > 0$ so that the only SVIC's that must be considered are $x_i, z \geq 0$ for $i = 1,2$. In the analysis presented here, however, we will assume that $x_i, z > 0$.

The optimal fire-support policy is shown in Figure 8 for the special case in which $z^f > 0$ and enemy survivors are valued in direct proportion to the rate at which they (i.e. Y_1 and Y_2) destroy value of the friendly forces (i.e. (43) holds). Although it is not impossible for it to be optimal for W to divide his fire between X_i and Z or X_1, X_2, and Z for a finite interval of time, we intuitively feel that this is an unlikely situation. For the $X_1 - X_2$ split, we have

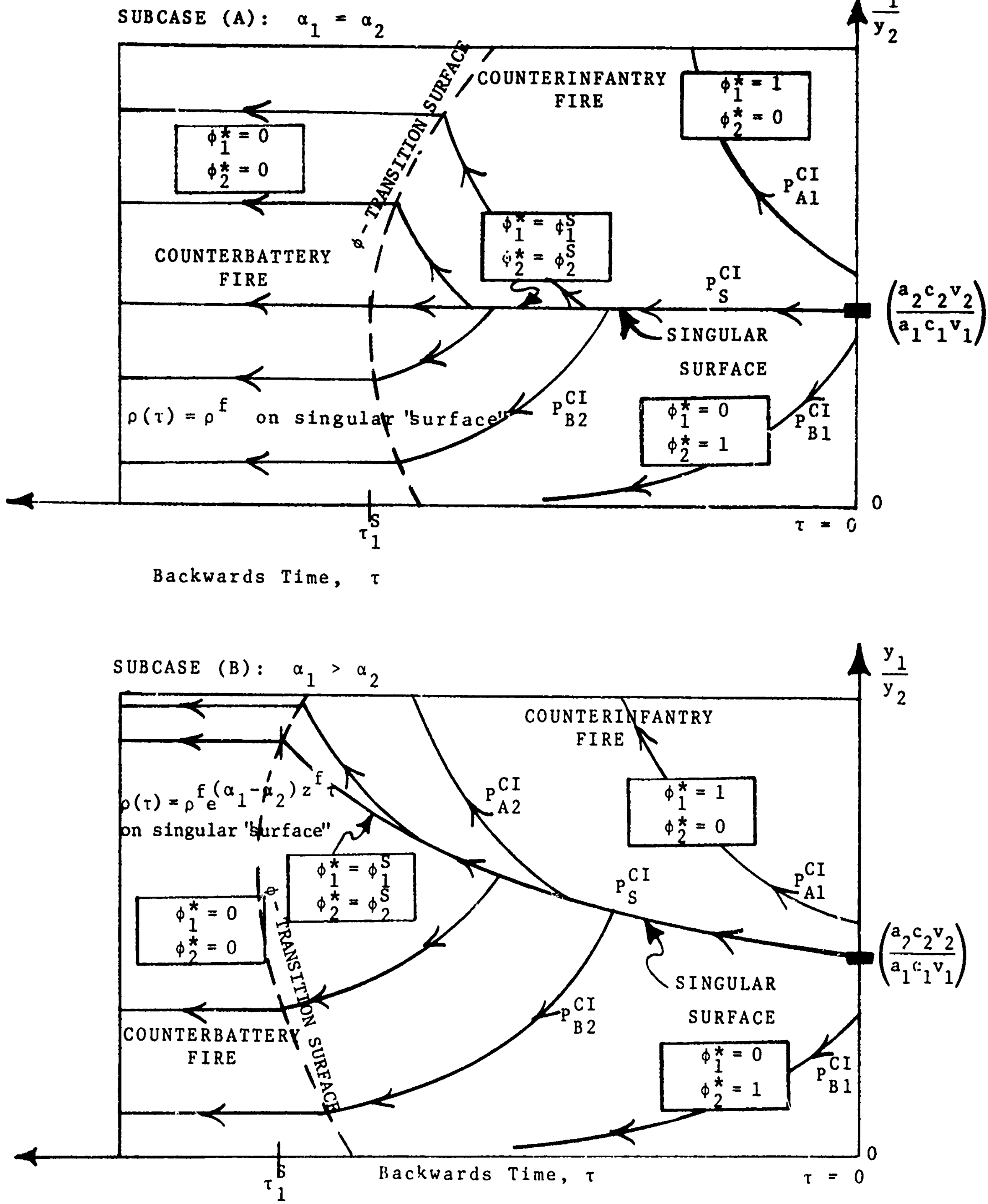

Figure 8. Diagram of Optimal (Closed-Loop) Fire-Support Policy for Problem 7 when $z^f > 0$ and $w_i = ka_iv_i$ for $i = 1,2$ (<u>not drawn to scale</u>).

$$y_1/y_2 = a_2c_2p_2/(a_1c_1p_1),$$

and

$$(-q_1)/(a_1p_1) = (-q_2)/(a_2p_2),$$

with the singular control given by

$$\phi_1^S = c_2/(c_1 + c_2) + (\alpha_1 - \alpha_2)z/(c_1 + c_2),$$

$$\phi_2^S = c_1/(c_1 + c_2) + (\alpha_2 - \alpha_1)z/(c_1 + c_2), \tag{81}$$

$$0 < \phi_1^S , \phi_2^S < 1 .$$

The switching time τ_1^S shown in Figure 8 is given by $K_i(\tau = \tau_1^S) = 0$, where

$$K_i(\tau) = a_ic_iv_iy_i^f \frac{1}{\theta}(e^{\theta\tau} - 1) - \beta\left(\sum_{k=1}^{2}\alpha_kv_kx_k^f\right)\tau$$

$$- \frac{\beta}{\theta^2} (e^{\theta\tau} - 1 - \theta\tau)\left(\sum_{k=1}^{2}\alpha_ka_kv_ky_k^f\right) + c_iw_iy_i^f . \tag{82}$$

We also have

$$\overset{\circ}{\rho} = (\phi_1c_1 - \phi_2c_2)\rho ,$$

where $\rho = y_1/y_2$. Then on the singular subarc (corresponding to the $X_1 - X_2$ split) for $0 \le \tau < \tau_1^S$, we have

$$\rho(\tau) = \rho^f e^{(\alpha_1 - \alpha_2)z^f\tau} .$$

6.10. Problem 8.

In the previous case (Problem 7) if the W fire-support units cause "linear-law" attrition of the Z fire-support units, then we obtain Problem 8 (see (1) and Table I). In this case the optimal time-sequential fire-support policy for

W may consist of dividing supporting fires between 2 or more of the enemy units. Details will be given in the future.

6.11. Problems 9 and 10.

These are analogues (see Table I) of Problems 1 and 2 in which the W fire-support units cause attrition to Y_i according to a "square-law" process (see discussion of such an assumption in Section 6.7). In this case, it is never optimal (assuming, for example in Problem 9, that $a_1b_1 \neq a_2b_2$) for the W fire-support units to split their fire between Y_1 and Y_2 for a finite interval of time. Details will be given in the future.

7. Consideration of Suppressive Effects of Supporting Weapsons.

Suppression may be considered to be the neutralization of a target (i.e. degradation of its combat capability) without actually destroying it (i.e. non-lethal effects)(see [17]). Although the objective of many fire-support missions (see [17] for further references) is suppression, a major deficiency in evaluating supporting weapons has been lack of mathematical models† of suppressive effects [17]. In this section we will briefly sketch some models which consider suppression and provide insights into optimal time-sequential fire-support allocation. These initial results are of a preliminary nature and hopefully will be refined in the future. The modeling of suppressive effects has been referred to as being in an "embryonic" state (p. 7 of [17]).

Suppression may be modeled either descriptively or prescriptively. In general, two ways to model suppressive effects within the context of Lanchester-type formulations are:

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†This includes development of a scientifically valid operational definition of suppression.

(a) modify Lanchester attrition-rate coefficients to reflect degraded fire effectiveness as more firers become suppressed,[†]

(b) consider combatants of a given class to be in different states (in the simplest model there are two states: unsuppressed and suppressed) with different fire effectiveness and vulnerability to enemy fire in each state; this approach requires model of state transitions.

Accordingly, for purposes of fire-support allocation two ways of modeling suppressive effects in Lanchester-type formulations are:

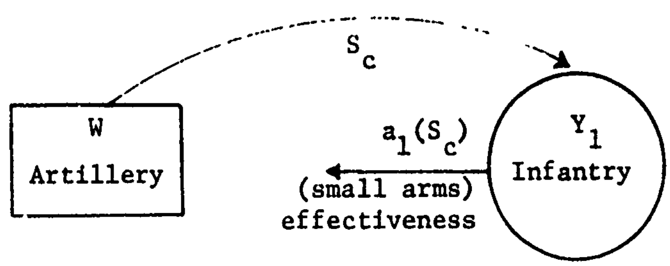
(A) degrade effectiveness of enemy fire as a function of fire effectiveness of friendly supporting weapons,

(B) give combatant choice as to his state (posture); in simplest model, there are two states (unsuppressed and suppressed) with the combatant in the unsuppressed state being both more effective in his fire and more vulnerable to enemy supporting fires.

We will briefly sketch[††] work along each of these lines in the next two sections.

7.1. Suppression Modeled Descriptively as a Degradation of Combat Effectiveness.

We consider a simple model (see p. 470 of [28]) for the suppressive effects of supporting fires. Let S_c denote the "lethality" of the W supporting fires.



[†] A more sophisticated approach would be to also modify the appropriate Lanchester attrition-rate coefficients to reflect decreased vulnerability of suppressed combatants.

[††] Details of the development of optimization results are omitted.

For a "linear-law" attrition process with a constant number of W units, constant rate of fire, etc., we have $S_c = c_1 y_1$. Assuming that suppression causes reduction in fire effectiveness and that suppression depends on the "density of lethality," we have

$$a_1 = a_1(c_1) .$$

If we assume that this functional dependence is linear, then

$$a_1 = a_1^o(1 - c_1/c_1^{sat}) ,$$

where c_1^{sat} denotes the level of supporting fire density at which the fire of Y_1 ceases to be effective.

Applying the above to the situation considered in Problem 2 (with $r_i(t) \equiv 0$), we obtain for $c_1 \leq c_2$ and $c_1^{sat} = c_2^{sat} = c_2$ the following:

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T) \right\} ,$$

with stopping rule: $t_f - T = 0$,

$$\text{subject to:} \quad \frac{dx_i}{dt} = -(1 - \alpha_i \phi_i) a_i y_i ,$$

$$\frac{dy_i}{dt} = -\phi_i c_i y_i \tag{83}$$

$$x_i, y_i \geq 0, \quad \phi_1 + \phi_2 = 1 , \quad \text{and} \quad \phi_i \geq 0 \quad \text{for} \quad i = 1,2,$$

where $0 < \alpha_1 \leq 1$ and $\alpha_2 = 1$ (i.e. $a_1 c_2 = c_1$). We use (3) and assume that $x_i(T) > 0$. Then when enemy forces are valued in direct proportion to the rate at which they destroy value of the friendly forces (i.e. (43) holds), the optimal fire-support policy is again POLICY A as given by (44). When

$w_1/(a_1v_1) > w_2/(a_2v_2)$, the optimal policy is as given in Table II with $F(\tau)$ of Table III being replaced by

$$F(\tau) = \tau + \left\{ \frac{1}{c_1} - \left(\frac{w_1}{a_1v_1} + \frac{1}{c_2} \right) \right\} e^{-c_1\tau} - \left\{ \frac{1}{c_1} - \left(\frac{w_2}{a_2v_2} + \frac{1}{c_2} \right) \right\} ,$$

and similarly for $G(\tau;\rho^f)$.

7.2. Suppression Modeled as a Rational Decision Process: A Differential Game Model.

In this section, we will consider suppression by supporting weapons to be the consequence of a rational decision process in which a combatant chooses his combat posture in order to "best" attain his combat objectives. Within the context of fire-support allocation, this may be formulated as a two-sided optimization process in which the friendly forces choose their time-sequential fire-support strategy and the enemy forces choose "posture strategies."

Let us again consider the situation of Problem 2 as shown in Figure 3. We consider each member of the Y_i force to be in either of two states: unsuppressed or suppressed. Let y_{i1} denote the number of Y_i that are unsuppressed and y_{i2} the number suppressed. Corresponding Lanchester attrition-rate coefficients are denoted as a_{ij} and c_{ij}. We assume that

$$a_{i2} = \alpha a_{i1} \quad \text{with} \quad 0 < \alpha < 1 ,$$

and

$$c_{i2} = \gamma c_{i1} \quad \text{with} \quad 0 < \gamma < 1 ,$$

i.e. a Y_i combatant in the suppressed state is both less effective in his fire against X_i and also less vulnerable to the W supporting fires. Then we may obtain the following differential game if we let W decide how to distribute his supporting fires over Y_1 and Y_2 and let Y_i choose the state he is in (i.e. either unsuppressed or suppressed):

$$\underset{\underset{\sim}{\Phi}}{\text{maximize}} \quad \underset{\underset{\sim}{\Psi}}{\text{minimize}} \left\{ \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k \left(\sum_{j=1}^{2} y_{kj}(T) \right) \right\},$$

with stopping rule: $t_f - T = 0$,

$$\text{subject to:} \quad \frac{dx_i}{dt} = -a_i(y_{i1} + \alpha y_{i2}) ,$$

$$\frac{dy_{i1}}{dt} = -\phi_i c_i y_{i1} + \psi_i , \tag{84}$$

$$\frac{dy_{i2}}{dt} = -\phi_i \gamma c_i y_{i2} - \psi_i ,$$

$$x_i, y_{ij} \geq 0, \quad \phi_1 + \phi_2 = 1, \quad \phi_i \geq 0 ,$$

$$\text{and} \quad -L_i \leq \psi_i \leq U_i \quad \text{for} \quad i = 1,2.$$

Although it is probably essentially impossible to obtain a complete solution to the differential game (84), this model does provide some valuable insights into when it is optimal for Y_i to be suppressed or not (giving consideration to the time-sequential fire-support strategy of W). Details will be given in the future. The theory of SVIC's (see [36]) is essential for solving this problem.

8. Discussion.

In this section we will review our above work on the determination of optimal time-sequential fire-support policies in several situations of tactical interest and discuss what we have learned about the dependence of these policies on the functional form of the combat attrition model. In our study, we have considered in various levels of detail a sequence of ten one-sided, time-sequential combat optimization problems (see Section 5). The conclusion given here are based on comparing and contrasting the solutions to these problems. We developed solutions to these problems by applying modern optimal control theory (see [6], [18], [24]) (assuming, if necessary, that no force level ever became zero†).

The results presented in this appendix are of a preliminary nature, being based on an initial examination. We have tried to consider many different problems and versions of problems, and this has led to incomplete results due to time constraints. We hope to refine such results in the future. We feel that this is an important and promising area of work and propose such further work to ONR as a future research task.

We saw that the optimal time-sequential fire-support policy could be quite complex for one of these problems, since if there were n types of forces on both sides, then the optimal (closed-loop) policy could depend on as many as $(n+1)$ state variables.†† This may be called the (usual) "curse of dimensionality." Thus, we saw the need for making approximations in order to simplify the optimal policy. We accordingly stressed some simple versions of our basic problem (e.g. Problems 2 and 3) and developed complete solutions in a few such cases and partial ones in others. We would propose to ONR as a future research task the completion of this program of solving the problems in the sequence of problems given in Section 5. When simplifying approximations are not made, it appears to be a straightforward (but messy) matter to develop a numerical solution by computational means. This may not be too convenient, however, if the optimal solution depends on a large number of state variables.

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†This assumtion is relaxed for problems considered in Appendix C. Breakpoints are considered for forces in these models there.

††Essentially, time behaved as an additional state variable in these fixed-length planning horizon problems.

We saw that the optimal time-sequential fire-support policy is strongly influenced by the nature of the combat attrition process. Indeed, the policy's basic structure is different in attack and defense situations. For example, for the enemy attack considered in Problem 3, it was never optimal to divide supporting fires between enemy attacking forces. When the friendly forces attacked (again with a "linear-law" attrition process of enemy infantry units by supporting fires), however, the optimal fire-support policy depended on enemy troop density, and it was sometimes optimal to split supporting fires between several enemy troop concentrations. As we had seen earlier [30], a "square-law" attrition process for enemy infantry units leads to concentration of supporting fires as an optimal policy, while a "linear-law" process can lead to splitting of fires. Thus, the determination of the appropriate mathematical description of the attrition processes (especially for supporting fires) appears to be an important task for future work (as well as estimation of attrition-rate coefficients).

We also obtained some insights into valuation of combat resources. When surviving enemy target types were valued in direct proportion to the rate at which they destroyed the value of friendly forces, we obtained a simple form for the optimal fire-support policy which was also very intuitively appealing. Such a simple optimal policy even resulted when there were temporal variations in the effectiveness of enemy defensive fires (reflecting, for example, the "closing" of one force with the other). We also examined optimal fire-support policies when suppressive effects of the supporting weapons were additionally considered. If enemy survivors were valued in direct proportion to their destruction of friendly value, then the optimal fire-support policy was exactly the same for a linear degradation of enemy fire effectiveness by supporting fires as for no suppressive effects. Otherwise, the structures of the optimal policies were similar, although computational investigations are needed.

Thus, we see that the optimal time-sequential allocation policy for supporting fires is strongly dependent on the mathematical nature of the attrition caused by these fires. The attrition process itself depends on such factors as target acquisition, command and control, battlefield intelligence, weapon system performance characteristics, tactical situation, etc. Additionally, all these determinations were made for a deterministic attrition process and perfect state information. The effects of uncertainty on the optimal fire-support policy should be investigated. We hope to investigate such aspects in the future.

9. Conclusions.

Based on the research reported in this appendix, we conclude that:

(1) an optimal time-sequential fire-support policy depends on the dynamics of combat and target priorities evolve dynamically over the course of battle,

(2) the nature of the attrition process (assumed to be Lanchester-type) for a supporting weapon system has a major influence on the structure of the optimal time-sequential fire-support policy as do those for other force interactions,

(3) the optimal time-sequential fire-support policy for an attack (approach to contact) is different in structure from that for the defense of such an attack,

(4) a "linear-law" attrition process for a supporting weapon system against enemy target types may lead to supporting fires being divided between enemy targets in an optimal time-sequential fire-support policy,

(5) a "square-law" attrition process always leads to concentration of fire on a single target as the optimal policy,

(6) judicious choice (i.e. valuation in direct proportion to their rate of destroying friendly value) of the value of enemy survivors (computed according to linear utilities) leads to a simple fire-support policy that is intuitively appealing; this policy remains optimal even when there are temporal variations in the affectiveness of enemy fire,

(7) simple "nearly optimal" fire-support policies may be developed through judicious approximations to the combat attrition process,

(8) if suppression is a linear function of the kill rate of the supporting weapon system, it has no effect on the optimal fire-support policy when enemy survivors are valued in direct proportion to their rate of destruction of friendly value (i.e. the optimal policy is not changed if the suppressive effects are excluded from the model).

REFERENCES

[1] H. Antosiewicz, "Analytic Study of War Games," Naval Res. Log. Quart. 2, 181-208 (1955).

[2] R. Bellman and S. Dreyfus, "On a Tactical Air-Warfare Model of Mengel," Opns. Res. 6, 65-78 (1958).

[3] L. Berkovitz and M. Dresher, "A Game Theory Analysis of Tactical Air War." Opns. Res. 7, 599-620 (1959).

[4] L. Berkovitz and M. Dresher, "Allocation of Two Types of Aircraft in Tactical Air War: A Game Theoretic Analysis," Opns. Res. 8, 694-706 (1960).

[5] H. Brackney, "The Dynamics of Military Combat," Opns. Res. 7, 30-44 (1959).

[6] A. Bryson and Y. C. Ho, Applied Optimal Control, Blaisdell Publishing Co., Waltham, Massachusetts, 1969.

[7] J. Busse, "An Attempt to Verify Lanchester's Equations," pp. 587-597 in Developments in Operations Research, Vol. 2, B. Avi-Itzhak (ed.), Gordon and Breach, New York, 1971.

[8] J. Engel, "A Verification of Lanchester's Law," Opns. Res. 2, 163-171 (1954).

[9] J. Funk and E. Gilbert, "Some Some Sufficient Conditions for Optimality in Control Problems with State Space Constraints," SIAM J. Control 8, 498-504 (1970).

[10] L. Giamboni, A. Mengel, and R. Dishington, "Simplified Model of a Symmetric Tactical Air War," The RAND Corporation, RM-711, August 1951.

[11] R. Isaacs, Differential Games, John Wiley & Sons, Inc., New York, 1965.

[12] R. Isaacs, "Differential Games: Their Scope, Nature, and Future," J. Opt. Th. Appl. 3, 283-295 (1969).

[13] C. Karns, "An Application of Lanchester's Equations to Amphibious Assaults," Study No. 1, Operations Research Group, Office of Naval Research, December 1953.

[14] A. Karr, "Stochastic Attrition Models of Lanchester Type," Paper P-1030, Institute for Defense Analysis, June 1974.

[15] Y. Kawara, "An Allocation Problem of Fire Support in Combat as a Differential Game," Opns. Res. 21, 942-951 (1973).

[16] H. Kelly, R. Kopp and H. Moyer, "Singular Extremals," pp. 63-101 in Topics in Optimization, G. Leitman (Ed.), Academic Press, New York, 1967.

[17] D. Kinney, "Modeling of Weapon Suppression Effects," NWC TP 5620, Naval Weapons Center, China Lake, California, February 1974.

[18] E. Lee and L. Markus, Foundations of Optimal Control Theory, John Wiley & Sons, Inc., New York, 1967.

[19] "Lulejian Artillery Model Study (DRAFT)," Lulejian & Associates, Inc., Torrance, California, June 1974.

[20] O. Mangasarian, "Sufficient Conditions for the Optimal Control of Nonlinear Systems," SIAM J. Control 4, 139-152 (1966).

[21] R. McNicholas and F. Crane, "Guide to Fire Support Mix Evaluation Techniques, Volume I: The Guide and Appendices A and B," Final Report, Naval Warfare Research Center, Stanford Research Institute, Menlo Park, California, March 1973.

[22] P. Morse and G. Kimball, Methods of Operations Research, the M.I.T. Press, Cambridge, Massachusetts, 1951.

[23] E. Paxon, "War Gaming," the RAND Corporation, RM-3489-PR, February 1963.

[24] L. Pontryagin, V. Boltyanskii, K. Gambrelidge, and E. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York, 1962.

[25] G. Pugh, "Theory of Measures of Effectiveness for General-Purpose Military Forces: Part II. Lagrange Dynamic Programming in Time-Sequential Combat Games," Opns. Res. 21, 886-906 (1973).

[26] G. Pugh and J. Mayberry, "Analysis of General-Purpose Force Procurement," Lambda Corporation, Paper 59, August 1971.

[27] G. Pugh and J. Mayberry, "Theory of Measures of Effectiveness for General-Purpose Military Forces: Part I. A Zero-Sum Payoff Appropriate for Evaluating Combat Strategies," Opns. Res. 21, 867-885 (1973).

[28] M. Schaffer, "Lanchester Models of Guerrilla Engagements," Opns. Res. 16, 457-488 (1968).

[29] J. Taylor, "On the Isbell and Marlow Fire Programming Problem," Naval Res. Log. Quart. 19, 539-556 (1972).

[30] J. Taylor, "Lanchester-Type Models of Warfare and Optimal Control," Naval Res. Log. Quart. 21, 79-106 (1974).

[31] J. Taylor, "Target Selection in Lanchester Combat: Linear-Law Attrition Process," Naval Res. Log. Quart. 20, 673-697 (1973).

[32] J. Taylor, "Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent Attrition-Rate Coefficients," Naval Res. Log. Quart. 21 683-704 (1974).

[33] J. Taylor, "Some Differential Games of Tactical Interest," Opns. Res. 22, 304-317 (1974).

[34] J. Taylor, "Survey on the Optimal Control of Lanchester-Type Attrition Process," to appear in Proceedings of the Conference on the State-of-the-Art of Mathematics in Combat Models (also Naval Postgraduate School Technical Report No. NPS-55Tw74031, March 1974).

[35] J. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems-Part II," Naval Postgraduate School Technical Report No. NPS-55Tw7211A, November 1972.

[36] J. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems-Part III," Naval Postgraduate School Technical Report No. NPS-55Tw74051, May 1974.

[37] J. Taylor, "Appendices C and D of 'Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems-Part III'," Naval Postgraduate School Technical Report No. NPS-55Tw74112, November 1974.

[38] J. Taylor, "On the Treatment of Force-Level Constraints in Time-Sequential Combat Problems," Naval Res. Log. Quart. 22, 617-650 (1975).

[39] C. Thomas and W. Deemer, "The Role of Operational Gaming in Operations Research," Opns. Res. 5, 1-27 (1957).

[40] USAF Assistant Chief of Staff, Studies and Analysis, "Methodology for Use in Measuring the Effectiveness of General Purpose Forces, SABER GRAND (ALPHA)," March 1971.

[41] H. Weiss, "Lanchester-Type Models of Warfare," pp. 82-98 in Proc. First International Conf. Operational Research, John Wiley and Sons, Inc., New York, 1957.

[42] H. Weiss, "Some Differential Games of Tactical Interest and the Value of a Supporting Weapon System," Opns. Res. 7, 180-196 (1959).

APPENDIX B: An Examination of the Effect of the Criterion Functional on the Optimal Fire-Support Policy

1. Introduction.

As we first pointed out in [16] (see also [18]), for the purposes of military analysis, it is convenient to consider that there are three essential parts of any time-sequential combat optimization problem:

(a) the decision criteria (for both combatants),

(b) the model of conflict termination conditions (and/or unit breakpoints),

(c) the model of combat dynamics.

It is important for the military analyst to understand the relationship between the nature of system objectives and the structure of optimal (time-sequential) combat strategies. Of particular importance is the sensitivity of the structure of optimal combat strategies to the nature of military objectives.† In a time-sequential combat optimization problem the combatant objectives are quantified through the criterion functional. If the optimal combat strategy (and associated payoff) were discovered to be quite sensitive to the functional form of the criterion functional, then one would know that great care must go into the selection of the functional form.

Pugh and Mayberry [11] have suggested†† that an appropriate payoff or objective function (in our terminology, criterion functional) for the quantitative evaluation of combat strategies is the loss ratio (possibly calculated using weighting factors for heterogeneous forces). They state [11] that an

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†See [11] for a discussion of the influence of political objectives on military objectives for the evaluation of (time-sequential) combat strategies.

††However, Pugh and Mayberry [11] do not explore the consequences of various functional forms for the criterion functional.

"almost equivalent" criterion is the loss difference. In this appendix we will examine to what extent this is true. In cases of either no replacements or a fixed-length planning horizon, it is easily seen that these criteria are equivalent to the ratio of survivors or the difference in survivors, respectively. It is such a case of no replacements that we will examine here.

Furthermore, it is of interest to consider the military worth (i.e. utility of military resources) of survivors. In almost all the work that has appeared in the open literature† a linear utility has been assumed for valuation of survivors, and some form of net military worth (i.e. the difference between the military worths of friendly and enemy survivors) has been taken as the payoff (i.e. criterion functional) (see, for example [9], [12]-[17], [20]).†† One reason for assuming such linear utilities is one of mathematical convenience: the boundary conditions for the dual variables do not depend on state variable values (at least when no system constraint involving the state variables is active).

The only systematic examinations of the influence of the nature of the criterion function on the structure of optimal time-sequential strategies known to the author are his own investigations [12]-[16], [19], [20]. In [12]-[16] a linear utility††† was assumed for the military worth of the numbers of

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†The only exception known to the author is the paper by Kawara [5] in which the payoff is the ratio of opposing infantry strengths (measured in terms of total numbers) at the "end of battle" (see also the differential game studied in Appendix D of [19]).

††A comprehensive review of pertinent literature prior to 1973 in the field of optimizing tactical decisions (using Lanchester-type models of warfare) is to be found in [17].

†††This means that the boundary conditions for the adjoint variables (at least when no state constraint is active) are independent of the values of the state variables. Serious computational difficulties <u>may</u> arise when a nonlinear utility is assumed. The effect of assuming a nonlinear utility for military resources upon the evaluation of time-sequential combat strategies has apparently never been studied.

each surviving weapon system type, and the criterion functional (payoff) was taken to be the net military worth of survivors (i.e. the difference between the military worths of friendly and of enemy forces). We then examined how the optimal time-sequential fire distribution policy depended on the assignment of these linear utilities in [12] through [16]. In other words, we examined the sensitivity of the optimal time-sequential combat policy to parametric variations in the assigned linear utilities for survivors. It was shown, for example, that the fire-distribution problems studied in [12]-[16] all possessed simple solutions when enemy survivors are valued in direct proportion to their kill capabilities (as measured by their Lanchester attrition-rate coefficients against the (homogeneous) friendly forces).

In [19] is the only study known to the author of the consequences of nonlinear utility for survivors. We determined (at least for the case in which the appropriate side's (in Kawara's case, the defender) supporting weapon system† is not annihilated) the most general form of the criterion functional which leads to optimal fire-support strategies being <u>independent</u> of force levels, and we showed that the criterion functional chosen by Kawara [5] is a special case of this. In other words, Kawara's conclusion [5] that optimal fire-support strategies do not depend on force levels only applies to problems with the special type of criterion functional used by Kawara and is not true in general. No other examination of the dependence of optimal strategies upon combatant objectives is known to the author.

Thus, the objective of the research reported in this appendix is to determine the sensitivity of the optimal time-sequential fire-support policy to the functional form of the criterion functional. Clearly, this must be

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†See [22] for a brief discussion of the distinction between a "primary" weapon system (or infantry) and a "supporting" weapon system.

examined for a concrete problem. Consequently, our research approach is to consider several different criterion functionals for the same tactical situation involving a time-sequential allocation of supporting fires. The tactical situation that we have chosen to examine is the "approach to contact" during an assault on enemy defensive positions by friendly ground forces. Additionally, we will consider an analytically tractable mathematical version of this problem (see Appendix A) so that we may make quantitative comparisons between the optimal policies corresponding to the various criterion functionals. Corresponding to each different criterion functional is a different optimization (here optimal control) problem. Each of these has been solved, and the corresponding optimal fire-support policies will be contrasted.

In this appendix three different criterion functionals are considered, and it is shown that the difference and the ratio of the military worths of friendly and enemy survivors (linear utilities) as criterion functionals may lead to exactly the same optimal policy. This is not true when we consider the weighted average of force ratios of opposing infantry at the time that the supporting fires are lifted as one of the criterion functionals. This objective leads to an essentially different type of optimal fire-support policy. We have decided that the two former criterion functionals (i.e. the difference and the ratio of military worths) are appropriate for an "attrition" strategy, whereas the weighted average of force ratios is appropriate for a "breakthrough" strategy. [In the latter case, the attacking force tries to overpower the defenders at one place along a front and then pour reinforcements through the breach in the defender's defenses in order to "penetrate" behind the enemy lines and disrupt enemy command, control, and communications.]

After carrying out the above research program, future research directions are suggested. We feel that it would be very worthwhile to extend the above study to cases of nonlinear utilities for survivors.

2. Comparison of Optimal Fire-Support Policies.

In this section we give the fire-support allocation problem for which the optimal policy is developed according to three different criterion functionals. These time-sequential fire-support policies are then compared.

2.1. The Fire-Support Problem.

Let us consider the attack of heterogeneous X forces against the static defense of heterogeneous Y forces along a "front." Each side is composed of primary units (or infantry) and fire-support units (or artillery). The X infantry (denoted as X_1 and X_2) launches an attack against the positions held by the Y infantry (denoted as Y_1 and Y_2). We may consider X_1 and X_2 to be infantry units operating on spatially separated pieces of terrain. We assume that the X_1 infantry unit attacks the Y_1 infantry unit and similarly for X_2 and Y_2 with no "crossfire" (i.e. the X_1 infantry is not attrited by the Y_2 infantry). We will consider only the "approach to contact" phase of the battle. This is the time from the initiation of the advance of the X_1 and X_2 forces towards the Y_1 and Y_2 defensive positions until the X_1 and X_2 forces actually make contact with the enemy infantry in "hand-to-hand" combat. It is assumed that this time is fixed and known to X.

The X_i forces begin their advance against the Y_i forces from a distance and move towards the Y_i position. The objective of the X_i forces during the "approach to contact" is to close with the enemy position as rapidly as possible. Accordingly, small arms fire by the X_i forces is held at a minimum or firing is done "on the move" to facilitate rapid movement. It is not unreasonable, therefore, to assume that the effectiveness of X_i fire "on the move" is negligible against Y_i.† We assume, however, that the defensive Y_i fire (for $i = 1,2$) causes attrition to the advancing X_i forces in their "field of fire" at a rate proportional to only the number of Y_i firers. Let a_i denote the constant of proportionality. It is convenient to refer to the attrition of a target type as being a "square-law" process when the casualty rate is proportional to the number of enemy firers only and as being a "linear-law" process when it is proportional to the product of the numbers of enemy firers and remaining targets. Brackney [1] has shown that a "square-law" attrition process occurs†† when the time to acquire targets is negligible in comparison with the time to destroy them. He points out that such a situation is to be expected to occur when one force assaults another. Additionally, we may consider the Y forces either to have no fire support units or that their fire support is "organic" to the Y units (i.e. fire support units are integrated with Y_i and only those with Y_i support Y_i).

During the "approach to contact" the X fire support units (denoted as W) deliver "area fire" against the Y_i forces.††† Let ϕ_i denote the fraction of the W fire support units which fire at Y_i. [We then have that $\phi_1 + \phi_2 = 1$ and $\phi_i \geq 0$ for $i = 1,2$.] Then for constant ϕ_i there are a constant number of fire support units firing at Y_i, since we assume that the

†It should be recalled that we have shown in Appendix A that such an approximation is necessary for reasons of mathematical tractability in the fire-support optimal control problem to be subsequently given.

††To be precise, one can only conjecture that such an attrition process probably occurs under the stated conditions.

†††In other words, we assume that X's fire support units fire into the (constant) area containing the enemy's infantry without feedback as to the destructiveness of this fire.

W fire support units are not in the combat zone and do not suffer attrition. In this case, the Y_i attrition rate is proportional to the Y_i force level (see [21]; also [4]). Let c_i denote the constant of proportionality. The combat situation is shown diagrammatically in Figure 1.

It is the objective of the X force to utilize their fire support units (denoted as W) over time in such a manner so as to achieve the "most favorable" situation at the end of the "approach to contact" at which time the force separations between opposing infantries are zero and artillery fires must be lifted from the enemy's positions in order not to also kill friendly forces. This "situation" or "outcome" may be measured in several different ways and is quantitatively expressed through the criterion functional (denoted as J). Thus, we have the following optimal control problem for the determination of the optimal time-sequential fire-support allocation policy (denoted as $\phi^*(t)$ for $0 \le t \le T$ where T denotes the time of the end of the "approach to contact") for the W fire-support units.

$$\underset{\phi_i(t)}{\text{maximize}}\ J,$$

$$\text{with stopping rule:} \quad t_f - T = 0,$$

$$\begin{array}{ll} \text{subject to:} & \dfrac{dx_i}{dt} = -a_i y_i, \\ \text{(battle dynamics)} & \dfrac{dy_i}{dt} = -\phi_i c_i y_i \quad \text{for} \quad i = 1,2, \end{array} \tag{1}$$

with initial conditions

$$x_i(t=0) = x_i^o \quad \text{and} \quad y_i(t=0) = y_i^o \quad \text{for} \quad i = 1,2,$$

and

$$x_1, x_2, y_1, y_2 \ge 0 \quad \text{(State Variable Inequality Constraints)}$$

$$\phi_1 + \phi_2 = 1 \quad \text{and} \quad \phi_i \ge 0 \quad \text{for} \quad i = 1,2 \quad \text{(Control Variable Inequality Constraints)},$$

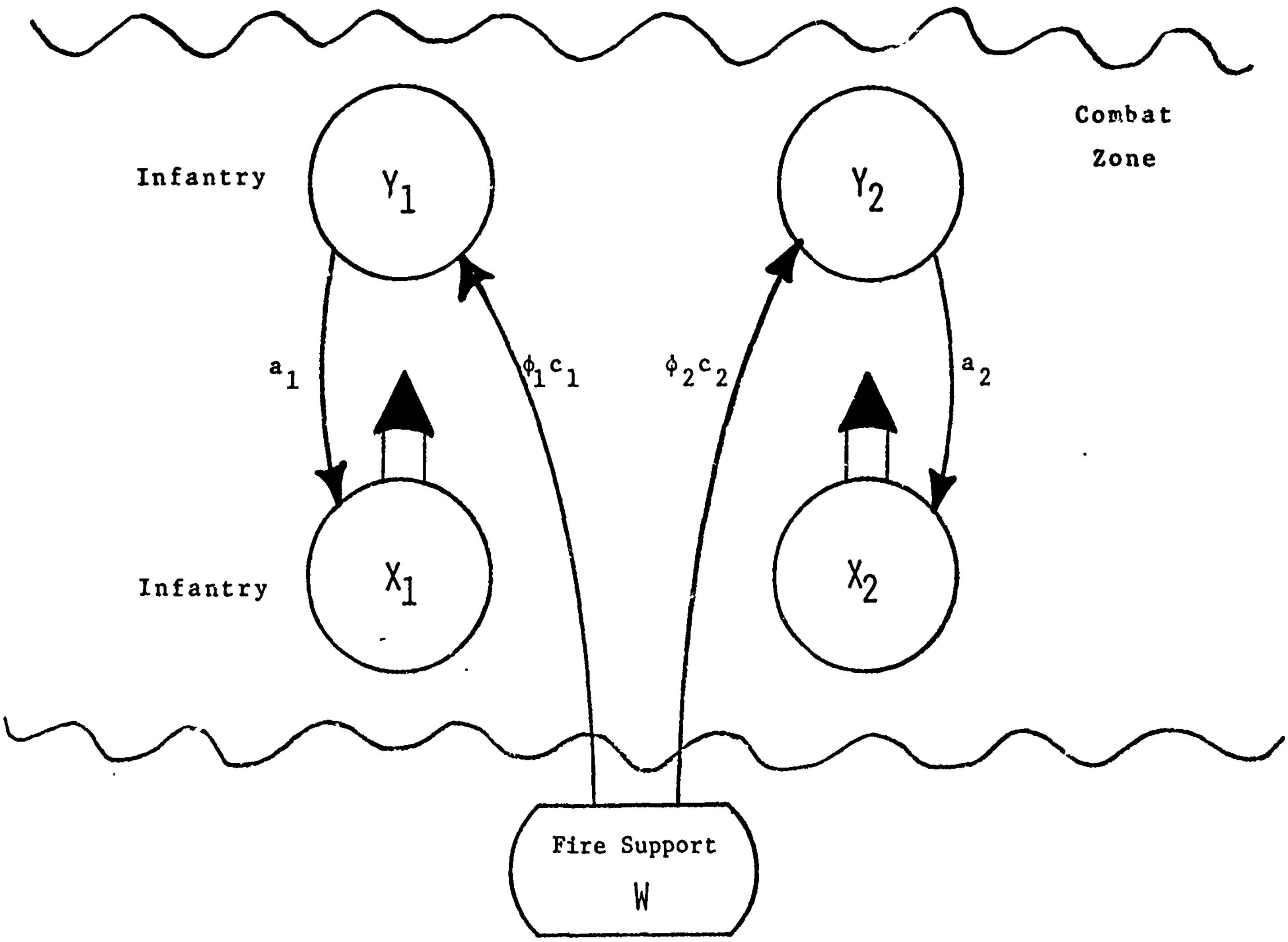

Figure 1. Diagram of Fire-Support Problem Considered for Examination of Effect of Criterion Function on Optimal Fire-Support Policy.

where

J denotes the criterion functional,

$x_i(t)$ denotes the number of X_i infantry at time t, similarly for $y_i(t)$,

a_i is a constant (Lanchester) attrition-rate coefficient (reflecting the effectiveness of Y_i fire against X_i),

c_i is a constant (Lanchester) attrition-rate coefficient (reflecting the effectiveness of W supporting fires against Y_i),

t_f (with numerical value T) denotes the end of the optimal control problem, and

ϕ_i denotes the fraction of W fire support directed at Y_i.

It will be convenient to consider the single control variable ϕ defined by

$$\phi = \phi_1 \text{ so that } \phi_2 = (1-\phi) \text{ and } 0 \le \phi \le 1. \tag{2}$$

It should be noted that for $T < +\infty$ it follows that we will always have $y_i(t) > 0$ for $i = 1,2$. Thus, the only state variable inequality constraints (SVIC's) that must be considered are $x_i \ge 0$. However, let us further assume that the attacker's infantry force levels are never reduced to zero. This may be militarily justified on the grounds that X would not attack the Y_i positions if his attacking X_i forces could not survive the "approach to contact." In Appendix C we consider some models in which this assumption is relaxed and breakpoints are considered for the various forces. Unfortunately, this leads to quite complex mathematical details.

2.2. Criterion Functionals Considered.

The three criterion functionals for which the optimal time-sequential fire-support allocation policies will be compared are given in Table I. All

Problem	Criterion Functional, J
1	$\sum_{k=1}^{2} \alpha_k x_k(T)/y_k(T)$
2	$\sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$
3	$\left(\sum_{k=1}^{2} v_k x_k(T)\right) / \left(\sum_{k=1}^{2} w_k y_k(T)\right)$

Table I. Summary of Problems Considered to Study Effect of Criterion Functional on Optimal Fire-Support Policy.

are functions only of the various numbers of combatants at the end of the planning horizon (i.e. at the end of the "approach to contact" at which time the supporting fires must be lifted for safety reasons).

The criterion functional for Problem 1 (i.e. $J_1 = \sum_{k=1}^{2} \alpha_k x_k(T)/y_k(T)$) represents a weighted average of the force ratios of opposing numbers of infantry in the two infantry combat zones. The rationale behind this is that in each combat area (i.e. the area of combat between X_i and Y_i) combat (possibly hand-to-hand) between the X_i and Y_i forces will follow the "approach to contact" and the (initial) force ratio will be related to the outcome of this subsequent combat action. The weighting factors α_k allow one to assign relative weights to this combat between X_i and Y_i in the two combat areas.

The criterion functional for Problem 2 (i.e. $J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$) represents the difference between the military worths (computed using a linear utility for survivors) of the X and Y forces, whereas the one for Problem 3 (i.e. $J_3 = \left(\sum_{k=1}^{2} v_k x_k(T)\right) / \left(\sum_{k=1}^{2} w_k y_k(T)\right)$) represents the ratio of military worths. Both these functionals have been proposed† by Pugh and Mayberry [11] as appropriate payoffs for the evaluation of combat strategies and have been said to be "almost equivalent" (see p. 869 of [11]).

2.3. Optimal Fire-Support Policies.

In this section we give the optimal time-sequential fire-support policies for the three problems†† stated in the previous section. In all cases we assume that neither of the attacking infantry forces can be reduced to a zero force

†Actually, Pugh and Mayberry [11] talk in terms of losses. See Section 1 above for a further discussion of this point.

††As shown in Table II, each of these problems corresponds to a different criterion functional for the attackers.

level during the approach to contact.[†] Under this condition the solutions[††] to the three problems are given in Table II with ancillary information on switching times being given in Table III.

Let us sketch here the proofs of a few statements made in Tables II and III. The existence of a unique nonnegative root to $F(\tau=\tau_S) = 0$ for $w_1/(a_1v_1) \geq w_2/(a_2v_2)$ follows from $F(\tau=0) \leq 0$ and $F'(\tau) > 0 \; \forall \; \tau \geq 0$. The existence of two positive roots to $G(\tau=\tau_\phi;\rho^f) = 0$ [here the second argument, ρ^f, is a (fixed) parameter] for $w_1/(a_1v_1) \geq w_2/(a_2v_2)$ and $\rho_L < \rho^f < \rho^f_S$ follows from $G(\tau=0) > 0$ for $\rho^f > \rho_L$ and the fact that (letting $\tilde{\tau}$ denote the (unique) value of τ at which the global minimum of the strictly convex function $G(\tau)$ occurs) $G(\tau=\tilde{\tau};\rho^f) = F(\tilde{\tau}) < 0$ for $\rho^f < \rho^f_S$. The latter is a consequence of $\frac{\partial G}{\partial \rho^f} > 0$ and $G(\tau=\tau_S;\rho^f=\rho^f_S) = F(\tau=\tau_S) = 0$. It should be noted that the fact that $G'(\tau=\tilde{\tau};\rho^f) = 0$ allows the parameter ρ^f to be eliminated from $G(\tau=\tilde{\tau};\rho^f)$. It also follows that there is no solution (i.e. value of τ_ϕ) to $G(\tau=\tau_\phi;\rho^f) = 0$ for $\rho^f > \rho^f_S$. The proof that $\frac{\partial \tau_S}{\partial J_3} = -\left(\frac{\partial F}{\partial J_3}\right)/\left(\frac{\partial F}{\partial \tau_S}\right) > 0$ follows from $\frac{\partial F}{\partial \tau_S} > 0$ and $\frac{\partial F}{\partial J_3} < 0$ (the latter holding since $(e^{-c_1\tau}-1+c_1\tau) > 0$).

For Problem 1 it is convenient to introduce the "local" force ratio $r_i = x_i/y_i$, which represents the ratio of the numbers of opposing infantry in each of the two combat areas (see Figure 1). The optimal time-sequential fire-support policy is most conveniently expressed as an open-loop control in terms

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[†]Initial force levels and the known length of the approach to contact may be sufficient to guarantee this for a given set (or range of values) of Lanchester attrition-rate coefficients.

[††]For a discussion of the distinction between open-loop and closed-loop time-sequential policies, see [16] or [20]. For deterministic models such as the ones under consideration, the two types of policies are well known to be equivalent.

Table II. Optimal Fire-Support Policies for the Three Problems.[†]

PROBLEM 1: $$J_1 = \sum_{k=1}^{2} \alpha_k x_k(T)/y_k(T)$$

For $0 \le t \le T$, optimal (open-loop) time-sequential fire-support policy is

$$\phi^*(t;r_1^o,r_2^o,T) = \begin{cases} 1 & \text{for } F_1(r_1^o,T) \ge F_2(r_2^o,T), \\ 0 & \text{for } F_1(r_1^o,T) \le F_2(r_2^o,T), \end{cases}$$

where

$$r_i = x_i/y_i,$$

and

$$F_i(r_i^o,T) = \alpha_i a_i c_i \left\{ \left(\frac{r_i^o}{a_i}\right)\left(\frac{e^{c_iT}-1}{c_i}\right) - \frac{1}{c_i^2}\left(e^{c_iT}-1-c_iT\right)\right\}.$$

PROBLEM 2: $$J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$$

and

PROBLEM 3: $$J_3 = \left(\sum_{k=1}^{2} v_k x_k(T)\right) \Big/ \left(\sum_{k=1}^{2} w_k y_k(T)\right)$$

Nonrestrictive Assumption: $w_1/(a_1v_1) \ge w_2/(a_2v_2)$

Optimal (closed-loop) time-sequential fire-support policy is

PHASE I for $0 \le t < t_1 = T - \tau_1(y_1^f/y_2^f)$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for } y_1/y_2 > a_2c_2v_2/(a_1c_1v_1), \\ c_2/(c_1+c_2) & \text{for } y_1/y_2 = a_2c_2v_2/(a_1c_1v_1), \\ 0 & \text{for } y_1/y_2 < a_2c_2v_2/(a_1c_1v_1), \end{cases}$$

PHASE II for $T - \tau_1(y_1^f/y_2^f) \le t \le T$

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = 1,$$

where

$$\tau_1 = \begin{cases} \tau_S & \text{for } \rho^f \ge \rho_S^f, \\ \tau_\phi & \text{for } \rho_L \le \rho^f < \rho_S^f, \\ 0 & \text{for } \rho^f < \rho_L, \end{cases}$$

$$\rho = y_1/y_2, \quad \text{and} \quad \rho_L = \left(\frac{a_2c_2v_2}{a_1c_1v_1}\right)\left(\frac{w_2}{a_2v_2}\right) \Big/ \left(\frac{w_1}{a_1v_1}\right).$$

NOTES:[††]

(1) τ_S is the unique nonnegative root of $F(\tau=\tau_S) = 0$.

(2) For $\rho_L < \rho^f < \rho_S^f$, τ_ϕ is the smaller of the two positive roots of $G(\tau=\tau_\phi;\rho^f) = 0$.

[†]It is assumed that problem parameters and initial force levels are such that $x_i(T) > 0$ for $i = 1,2$.

[††]See Table III for the definitions of $F(\tau)$ and $G(\tau;\rho^f)$. These functions are different for Problems 2 and 3.

Table III. Determination of the Switching Times τ_S and τ_ϕ for Problems 2 and 3.

Nonrestrictive Assumption: $w_1/(a_1v_1) \geq w_2/(a_2v_2)$

τ_S is the unique nonnegative root of $F(\tau=\tau_S) = 0$. For $\rho_L < \rho^f < \rho_S^f$, τ_ϕ is the smaller of the two positive roots of $G(\tau=\tau_\phi;\rho^f) = 0$.

It has been shown that

(a) bounds on τ_ϕ are given by $0 \leq \tau_\phi < \tau_S$,

(b) τ_ϕ is a strictly increasing function of ρ^f for $\rho_L \leq \rho^f < \rho_S^f$,

(c) there is no root to $G(\tau=\tau_\phi;\rho^f) = 0$ for $\rho^f > \rho_S^f$.

For PROBLEM 2: $J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$

$$F(\tau) = \tau + \left(\frac{1}{c_1} - \frac{w_1}{a_1v_1}\right)e^{-c_1\tau} - \left(\frac{1}{c_1} - \frac{w_2}{a_2v_2}\right)$$

$$G(\tau;\rho^f) = \frac{1}{c_1}(e^{c_1\tau}-1)\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\rho^f - \tau + \left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\left(\frac{w_1}{a_1v_1}\right)\rho^f - \left(\frac{w_2}{a_2v_2}\right)$$

Bounds on τ_S are given by:

(a) For $w_1/(a_1v_1) \leq 1/c_1$,

$$\frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2} \leq \tau_S \leq \frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\}.$$

(b) For $1/c_1 \leq w_1/(a_1v_1)$,

$$\frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\} \leq \tau_S \leq \frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2}.$$

For PROBLEM 3: $J_3 = \left(\sum_{k=1}^{2} v_k x_k(T)\right)/\left(\sum_{k=1}^{2} w_k y_k(T)\right)$

$$F(\tau) = \tau + \left(\frac{1}{c_1} - \frac{J_3w_1}{a_1v_1}\right)e^{-c_1\tau} - \left(\frac{1}{c_1} - \frac{J_3w_2}{a_2v_2}\right)$$

$$G(\tau;\rho^f) = \frac{1}{c_1}(e^{c_1\tau}-1)\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\rho^f - \tau + J_3\left\{\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\left(\frac{w_1}{a_1v_1}\right)\rho^f - \left(\frac{w_2}{a_2v_2}\right)\right\}$$

Bounds on τ_S are given by:

(a) For $J_3w_1/(a_1v_1) \leq 1/c_1$,

$$J_3\left(\frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2}\right) \leq \tau_S \leq \frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\}.$$

(b) For $1/c_1 \leq J_3\, w_1/(a_1v_1)$,

$$\frac{1}{c_1}\left\{1 - \left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)\right\} \leq \tau_S \leq J_3\left(\frac{w_1}{a_1v_1} - \frac{w_2}{a_2v_2}\right).$$

Also

$$\frac{\partial \tau_S}{\partial J_3} > 0.$$

of the two initial force ratios, denoted as $r_i^o = r_i(t=0)$ for $i = 1,2$, and the known length of time for the approach to contact T. This optimal fire-support policy is graphically depicted in Figure 2. In the initial force-ratio space, the line with equation

$$r_2^o = R\gamma \frac{a_2}{a_1} r_1^o - \mu a_2, \tag{3}$$

where

$$R = \alpha_1 a_1 c_1/(\alpha_2 a_2 c_2),$$

$$\gamma = \left(\frac{c_2}{c_1}\right)\left(\frac{e^{c_1 T}-1}{e^{c_2 T}-1}\right),$$

and

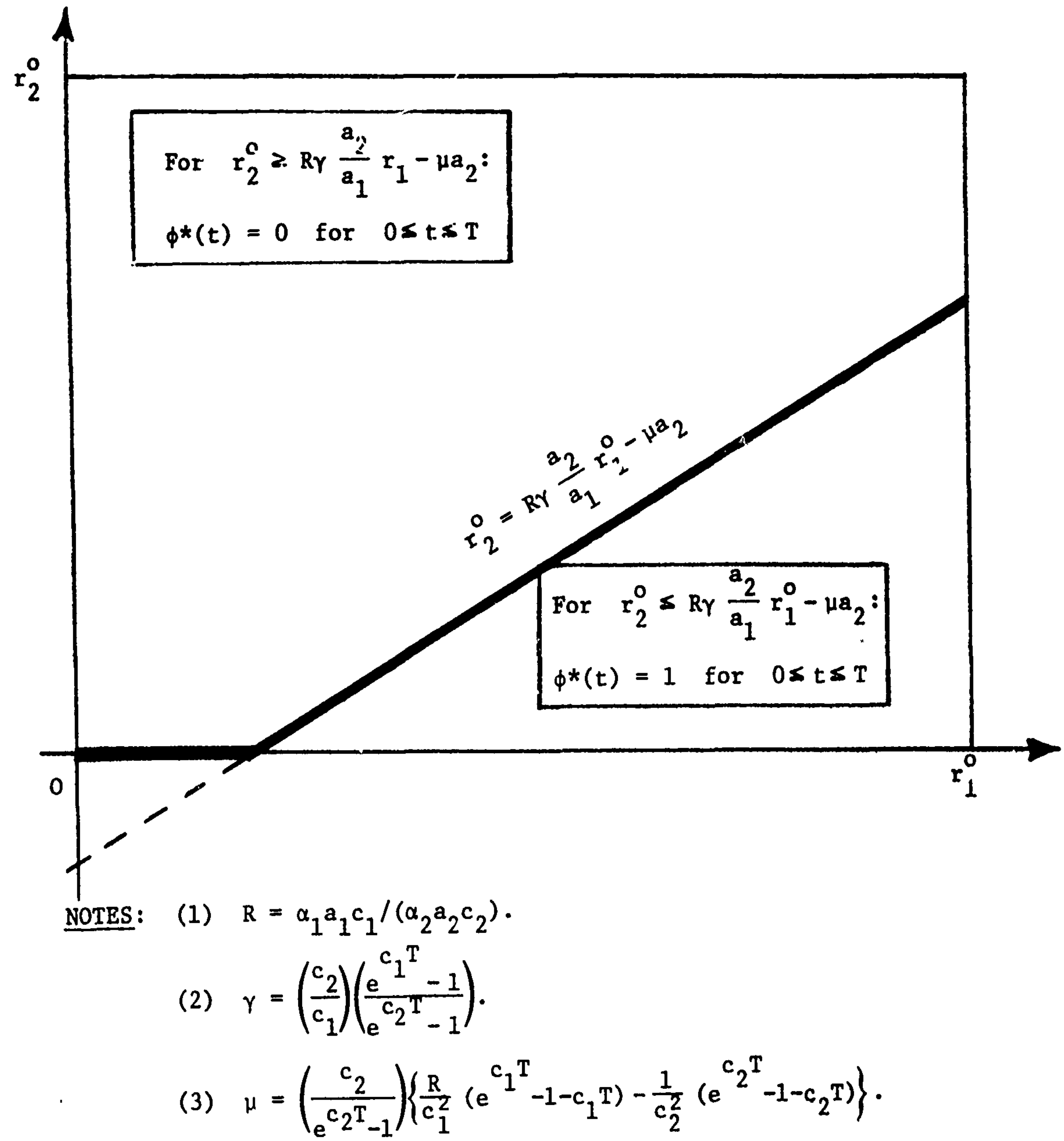
$$\mu = \left(\frac{c_2}{e^{c_2 T}-1}\right)\left\{\frac{R}{c_1^2}\left(e^{c_1 T}-1-c_1 T\right) - \frac{1}{c_2^2}\left(e^{c_2 T}-1-c_2 T\right)\right\},$$

is a "dispersal line" (see [12] or [16]) away from which all optimal battle trajectories flow. This is shown in Figure 3. In constructing this figure, use has been made of facts like the following: when $\phi = 1$ for $0 \le t \le T$ and $r_2^f = 0$, then

$$r_1 = \frac{1}{c_1}\left\{(c_1 r_1^f - a_1)e^{-c_1 r_2/a_2} + a_1\right\}. \tag{4}$$

For Problems 2 and 3, the optimal fire-support policy (expressed as a closed loop control (see [16] or [20])) is most conveniently expressed in terms of y_1/y_2 (i.e. the ratio of the numerical strengths of the two defending infantry forces) and $\tau = T - t$ (i.e. the "backwards" time or "time to go" in the approach to contact). When enemy forces are valued in direct proportion to the rate at which they destroy value of the friendly forces, i.e.

$$w_i = k a_i v_i \quad \text{for} \quad i = 1,2, \tag{5}$$



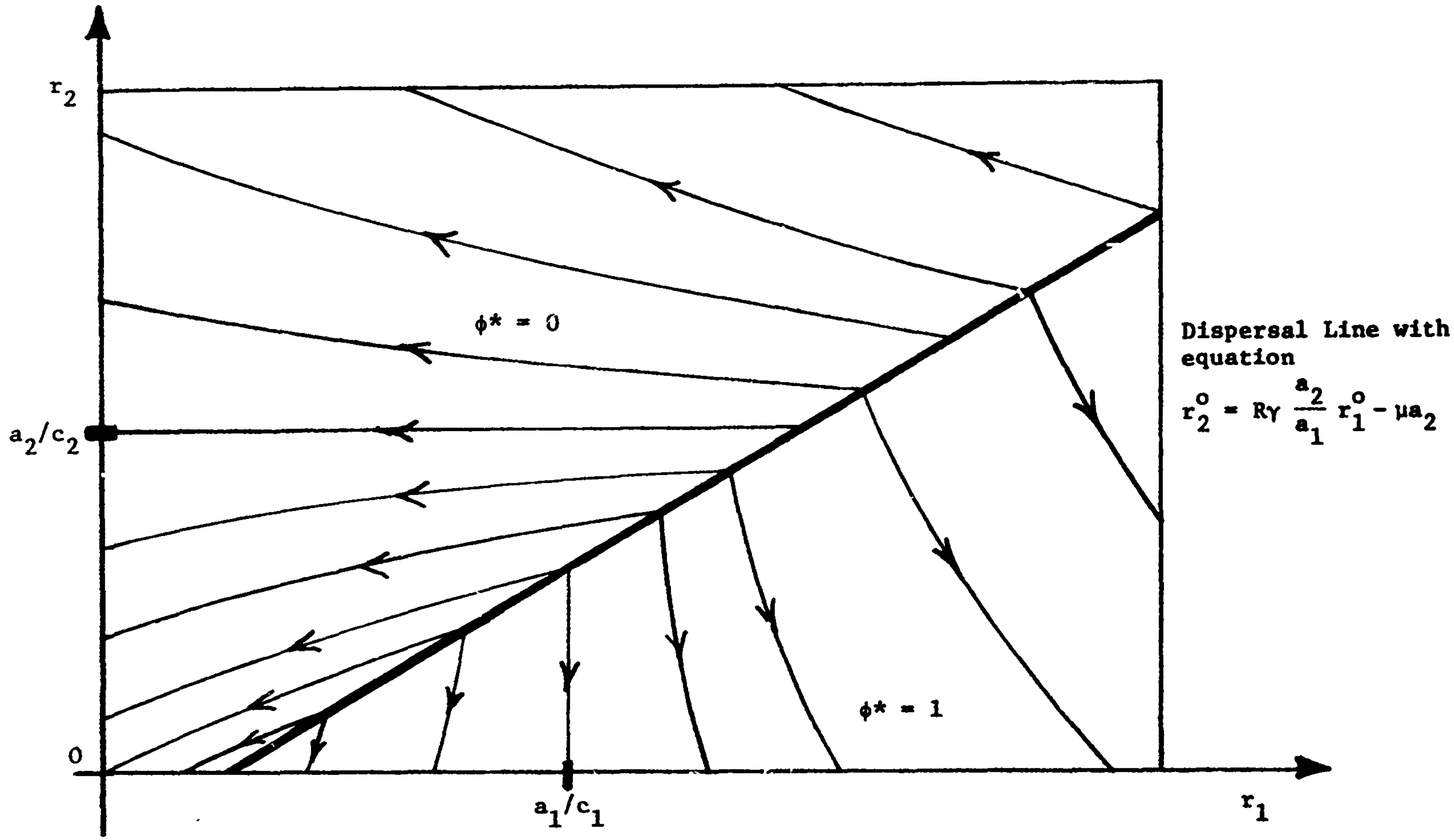
<u>NOTES</u>: (1) $R = \alpha_1 a_1 c_1/(\alpha_2 a_2 c_2)$.

(2) $\gamma = \left(\frac{c_2}{c_1}\right)\left(\frac{e^{c_1 T} - 1}{e^{c_2 T} - 1}\right)$.

(3) $\mu = \left(\frac{c_2}{e^{c_2 T} - 1}\right)\left\{\frac{R}{c_1^2}\,(e^{c_1 T} - 1 - c_1 T) - \frac{1}{c_2^2}\,(e^{c_2 T} - 1 - c_2 T)\right\}$.

(4) $r_i = x_i/y_i$.

Figure 2. Optimal (Open-Loop) Fire-Support Policy for PROBLEM 1.



NOTE: The definitions of the quantities R, γ, μ, and r_1 are given in Figure 2.

Figure 3. Optimal Battle Trajectories Resulting from Optimal (Open-Loop) Fire-Support Policy for PROBLEM 1 (not drawn to scale).

the optimal fire-support policy takes a particularly simple form (denoted as POLICY A):

POLICY A: For $0 \leq t \leq T$,

$$\phi^*(t,\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 1 & \text{for } y_1/y_2 > a_2c_2v_2/(a_1c_1v_1), \\ c_2/(c_1+c_2) & \text{for } y_1/y_2 = a_2c_2v_2/(a_1c_1v_1), \\ 0 & \text{for } y_1/y_2 < a_2c_2v_2/(a_1c_1v_1). \end{cases} \qquad (6)$$

This is shown pictorially in Figure 4 in which optimal trajectories are traced backwards in time. It is convenient to note that, for example, when $\phi(\tau) =$ CONSTANT for $0 \leq \tau \leq \sigma$, we have

$$\rho(\tau) = \rho^f \exp\{[\phi c_1-(1-\phi)c_2]\tau\}.$$

In this case, $\tau_1 = 0$ (see Table II), i.e. the entire approach to contact is "PHASE I."

When enemy forces are not valued in direct proportion to the rate at which they destroy value of the friendly forces (without loss of generality we may assume that $w_1/(a_1v_1) > w_2/(a_2v_2)$), the solutions to Problems 2 and 3 are considerably more complex as shown in Figure 5. As we see from Table II, the planning horizon may be considered to consist of two phases (denoted as PHASE I and as PHASE II) during each of which a different fire-support allocation rule is optimal. We denote this policy as POLICY B (see Table II). During PHASE I, POLICY A is optimal; whereas during PHASE II, it is optimal to concentrate all artillery fire on Y_1 (which has been valued disproportionately high). The absence or presence of PHASE II itself in the optimal time-sequential fire support policy depends on the ratio of enemy strengths $\rho = y_1/y_2$. For Problem 2 the length of PHASE II (i.e. τ_1) is <u>independent</u> of the final force levels of the attacking infantry units (i.e. x_1^f and x_2^f) but depends only on

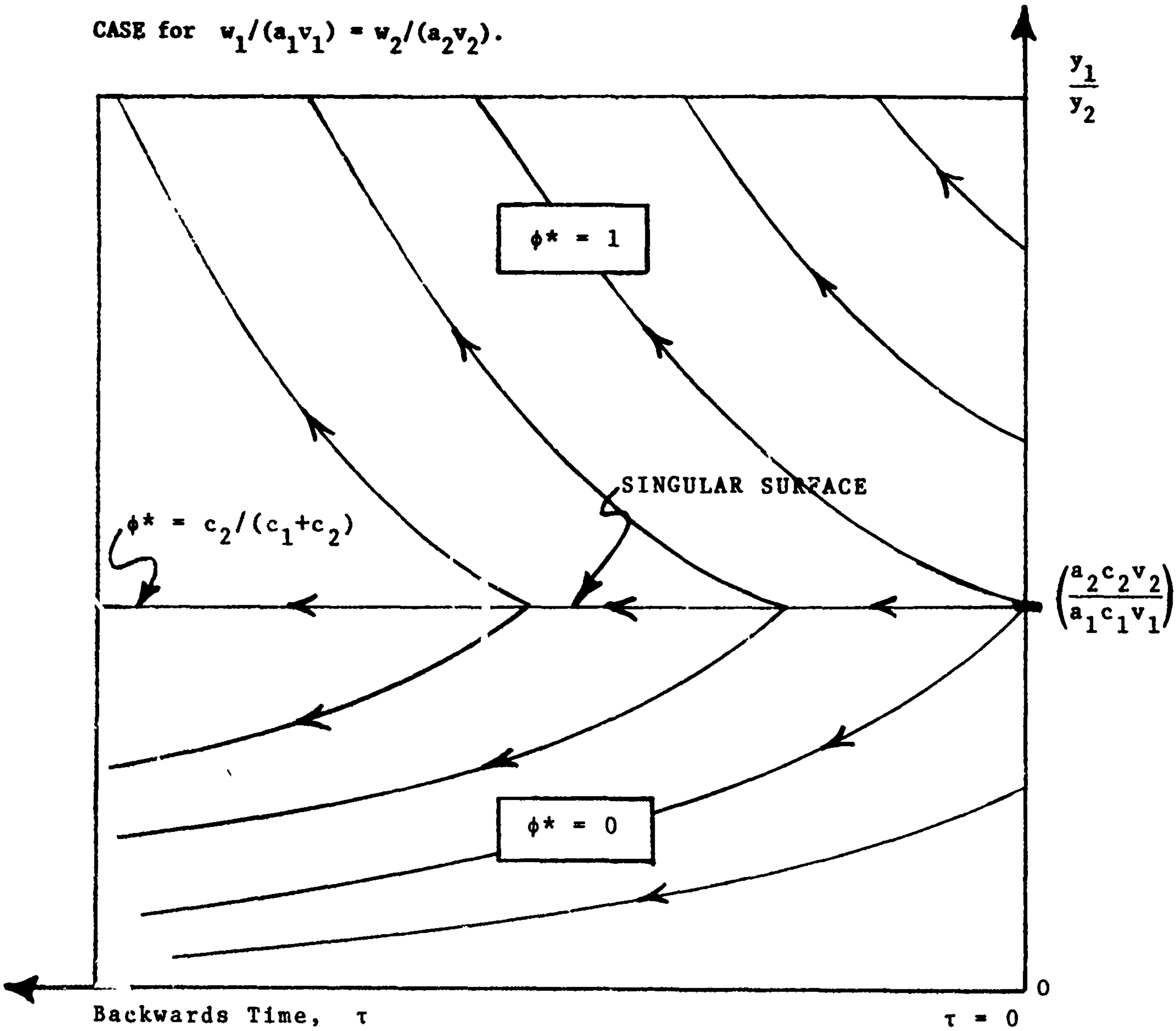
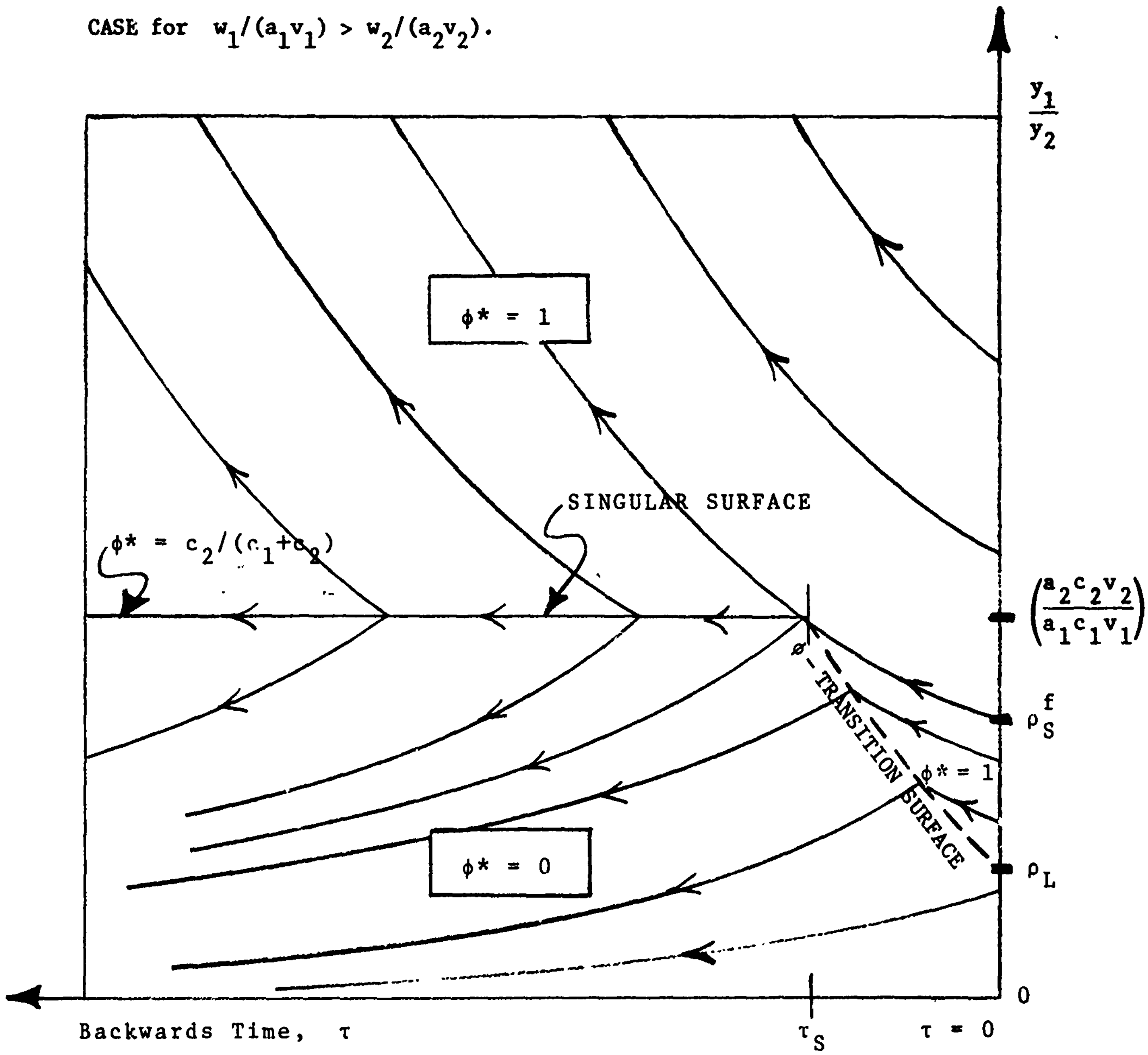


Figure 4. Diagram of Optimal (Closed-Loop) Fire-Support Policy (POLICY A) for PROBLEMS 2 and 3 when $w_1/(a_1v_1) = w_2/(a_2v_2)$ (not drawn to scale).

CASE for $w_1/(a_1v_1) > w_2/(a_2v_2)$.



NOTES: (1) $\rho = y_1/y_2$

(2) See Table II for definitions of ρ_L and ρ_S^f.

Figure 5. Diagram of Optimal (Closed-Loop) Fire-Support Policy (POLICY B) for PROBLEMS 2 and 3 when $w_1/(a_1v_1) > w_2/(a_2v_2)$ (<u>not drawn to scale</u>).

$\rho^f = y_1^f/y_2^f$ and the combat effectiveness parameters (see equations (1)), whereas for Problem 3 the length of PHASE II does depend directly on x_1^f and x_2^f through the criterion functional $J_3 = \sum_{k=1}^{2} v_k x_k^f / \sum_{k=1}^{2} w_k y_k^f$. Thus, we see that τ_1 may be quite different for Problems 2 and 3 when $w_1/(a_1 v_1) > w_2/(a_2 v_2)$. [At the time of the writing of this report we have not performed computational studies to compare values of τ_1 for these two problems.]

2.4. Discussion of Comparison.

In this section we will contrast the structures of the optimal time-sequential fire-support policies for the three problems considered above. Let us recall that we have assumed in all cases that $x_1^f, x_2^f > 0$.

For Problem 1 the optimal fire-support policy is to always concentrate all artillery fire (i.e. supporting fires) on one of the opposing enemy infantry units. This will maximize the force ratio (i.e. x_1^f/y_1^f) at the end of the approach to contact in one of the combat areas and may be considered to be a "breakthrough" tactic. In other words, one concentrates all fire support on the key enemy unit in order to overwhelm it and effect a penetration.

On the other hand, for Problems 2 and 3 the optimal fire-support policy may involve splitting of fires between the two enemy troop concentrations. This property of the solution has been anticipated in our earlier work on the optimal control of "linear-law" Lanchester-type attrition processes [13], [14] (see also [19]). We may consider this policy to be an "attrition" tactic which aims to wear down the overall enemy strength. The structures of the optimal policies for Problems 2 and 3 are similar, although the switching times (i.e. τ_ϕ and τ_S) may be appreciably different. [More computational work needs to be done here, and we hope to do this in the future.] The functional dependences of the switching times are also different. For Problem 2 the switching times (i.e. the ϕ-transition surface) are independent of the attacking force levels (i.e. x_1 and x_2) (as is the optimal policy itself), whereas for Problem 3 the switching times depend on the ratio of military worths (computed using linear utilities), i.e. $J_3 = \left(\sum_{k=1}^{2} v_k x_k(T)\right) / \left(\sum_{k=1}^{2} w_k y_k(T)\right)$. It has been shown (see Section 2.3 above) that $\frac{\partial \tau_S}{\partial J_3} > 0$ (although $\lim_{J_3 \to +\infty} \tau_S = \frac{1}{c_1} \ln\{(w_1/(a_1 v_1))/(w_2/(a_2 v_2))\}$) so that the larger that J_3 becomes, the more time is spent concentrating fire on Y_1.

The most significant thing to be noted in comparing the optimal fire support policies for these three problems is that the entire structure of the optimal policy is changed by changing the criterion functional. In particular, singular subarcs (i.e. the splitting of W's fire between Y_1 and Y_2) do not appear in the optimal policy for Problem 1, even though the necessary conditions for optimality on singular subarcs are exactly the same in all three of these problems. Such singular subarcs are, of course, part of the solution for Problems 2 and 3.

3. Development of Optimal Policy for Problem 1.

The optimal policy is developed by application of modern optimal control theory. For Problem 1 it is convenient to introduce the force ratio in the i^{th} combat zone $r_i = x_i/y_i$. Then Problem 1 may be written as

$$\underset{\phi_i(t)}{\text{maximize}} \sum_{k=1}^{2} \alpha_k r_k(T) \quad \text{with } T \text{ specified,}$$

$$\text{subject to: } \frac{dr_i}{dt} = -a_i + \phi_i c_i r_i \quad \text{for } i = 1,2, \tag{7}$$

$$\phi_1 + \phi_2 = 1, \quad \phi_i \geq 0, \quad \text{and} \quad r_i \geq 0 \quad \text{for } i = 1,2,$$

where we recall (2). We also recall that we have assumed that $r_i > 0$.

3.1. Necessary Conditions of Optimality.

The Hamiltonian [2] is given by (using (2))

$$H = \lambda_1(-a_1+\phi c_1 r_1) + \lambda_2(-a_2+(1-\phi)c_2 r_2), \tag{8}$$

so that the maximum principle yields the extremal control law

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad S_\phi(t) > 0, \\ 0 & \text{for} \quad S_\phi(t) < 0, \end{cases} \tag{9}$$

where $S_\phi(t)$ denotes the ϕ-switching function defined by

$$S_\phi(t) = c_1\lambda_1 r_1 - c_2\lambda_2 r_2. \tag{10}$$

The adjoint system of equations (again using (2) for convenience) is given by (assuming that $r_i(T) > 0$)

$$\dot{\lambda}_i = -\phi_i^* c_i \lambda_i \quad \text{with} \quad \lambda_i(T) = \alpha_i \quad \text{for} \quad i = 1,2. \tag{11}$$

Computing the first two time derivatives of the switching function

$$\dot{S}_\phi(t) = -a_1c_1\lambda_1 + a_2c_2\lambda_2, \tag{12}$$

$$\ddot{S}_\phi(t) = a_1c_1\lambda_1(c_1\phi) - a_2c_2\lambda_2(c_2(1-\phi)), \tag{13}$$

we see that on a singular subarc† we have [2], [6]

$$r_1/a_1 = r_2/a_2, \tag{14}$$

$$a_1c_1\lambda_1 = a_2c_2\lambda_2, \tag{15}$$

with the singular control given by

$$\phi_S = c_2/(c_1+c_2). \tag{16}$$

On such a singular subarc the generalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1c_1\lambda_1(c_1+c_2) > 0$.

†See [14] for a further discussion.

3.2. Synthesis of Extremals.

In synthesizing extremals† by the usual backwards construction procedure (see, for example, [12] or [14]) it is convenient to introduce the "backwards" time τ defined by $\tau = T - t$. Rather than explicitly constructing extremals and determining domains of controllability (see [12], [16], [20]), it is more convenient to show that the return (i.e. value of the criterion functional) corresponding to certain extremals dominates that from others. For this purpose it suffices to determine all possible types of extremal policies as we will now do.

We then have that

$$S_\phi(\tau=0) = \alpha_2 a_2 c_2 (R r_1^f / a_1 - r_2^f / a_2), \tag{17}$$

where

$$R = \alpha_1 a_1 c_1 / (\alpha_2 a_2 c_2). \tag{18}$$

Without loss of generality we may <u>assume that $R \geq 1$</u>. Then by (12) we have

$$\mathring{S}_\phi(\tau=0) = \alpha_1 a_1 c_1 - \alpha_2 a_2 c_2 \geq 0, \tag{19}$$

where $\mathring{S}_\phi$ denotes the "backwards" time derivative $\mathring{S}_\phi = dS_\phi/d\tau$. Considering (12) we may write

$$\mathring{S}_\phi(\tau) = \alpha_2 a_2 c_2 \{R(\lambda_1/\alpha_1) - (\lambda_2/\alpha_2)\}. \tag{20}$$

It follows that $S_\phi(\tau) > 0$ and $\phi^*(\tau) = 1 \ \forall \ \tau > 0$ when $S_\phi(\tau=0) \geq 0$ for $R > 1$ (also when $S_\phi(\tau=0) > 0$ for $R = 1$). We also have $S_\phi(\tau) < 0$ and $\phi^*(\tau) = 0 \ \forall \ \tau \geq 0$ when $S_\phi(\tau=0) < 0$ for $R = 1$.

There may be a change in sign of $S_\phi(\tau)$ when $S_\phi(\tau=0) < 0$ for $R > 1$. In this case $\phi^*(\tau) = 0$ for $0 \leq \tau \leq \tau_1$ and then

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† By an extremal we mean a trajectory on which the necessary conditions of optimality are satisfied.

$$S_\phi(\tau) = \alpha_2 a_2 c_2 \{R r_1(\tau)/a_1 - e^{c_2\tau} r_2(\tau)/a_2\}, \tag{21}$$

where τ_1 denotes the smallest value of τ such that $S_\phi(\tau=\tau_1) = 0$. It is clear that we must have $\dot{S}_\phi(\tau=\tau_1) \geq 0$. If $\mathring{S}_\phi(\tau=\tau_1) > 0$, then we have a <u>transition surface</u>, and from (21) we find that

$$R r_1(t_1)/a_1 - e^{c_2\tau_1} r_2(t_1)/a_2 = 0, \tag{22}$$

where $t_1 = T - \tau_1$. From (20) we find that

$$0 \leq \tau_1 < \frac{1}{c_2} \ln R. \tag{23}$$

If $\mathring{S}_\phi(\tau=\tau_1) = 0$, the <u>singular subarc</u> may be entered, and then we have

$$\tau_1 = \frac{1}{c_2} \ln R. \tag{24}$$

In this case we have

$$r_2^f = R r_1^f a_2/a_1 + F(R) a_2/c_2, \tag{25}$$

where $r_i^f = r_i(t=T)$ and $F(R) = 1 + R(\ln R - 1)$. We easily see that $F(R) > 0$ for $R > 1$. When $R = 1$ we see that once the singular subarc is entered (in forwards time), it is never exited by an extremal trajectory.

For the purposes of determining the optimal policy it suffices to consider the following four extremal policies.

$$\text{Policy 0:} \qquad \phi^*(t) = 0 \quad \text{for} \quad 0 \leq t \leq T, \tag{26}$$

$$\text{Policy 1:} \qquad \phi^*(t) = 1 \quad \text{for} \quad 0 \leq t \leq T, \tag{27}$$

$$\text{Policy B-B:} \qquad \phi^*(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t < T - \tau_1, \\ 0 & \text{for} \quad T - \tau_1 \leq t \leq T, \end{cases} \tag{28}$$

where $0 \leq \tau_1 < \frac{1}{c_2} \ln R$, and†

$$\text{Policy S:} \qquad \phi^*(t) = \begin{cases} c_2/(c_1+c_2) & \text{for} \quad 0 \leq t < T - \tau_1, \\ 0 & \text{for} \quad T - \tau_1 \leq t \leq T, \end{cases} \tag{29}$$

---

†The only extremal policies that are omitted here are those corresponding to extremals which contain a singular subarc but $r_1^o/a_1 \neq r_2^o/a_2$.

where $\tau_1 = \frac{1}{c_2} \ln R$ and $r_1^o/a_1 = r_2^o/a_2$. It is readily seen from (17) that Policy 0 yields $Rr_1^f/a_1 \geq r_2^f/a_2$, etc. We also note that corresponding to the bang-bang policy (28) we have

$$\begin{aligned} r_1(t_1) &= \{(c_1r_1^o - a_1)e^{c_1t_1} + a_1\}/c_1, \\ r_2(t_1) &= r_2^o - a_2t_1 \geq 0. \end{aligned} \tag{30}$$

3.3. Determination of the Optimal Fire-Support Policy.

As we have discussed elsewhere [13]-[16], [20], the optimality of an extremal trajectory may be proven by citing the appropriate existence theorem for an optimal control to the problem at hand; there are two further subcases: (1) if the extremal is unique, then it is optimal or (2) if the extremal is not unique and only a finite number exist, then the optimal trajectory is determined by considering the finite number of corresponding values of the criterion functional.† The existence of a measurable optimal control follows by Corollary 2 on p. 262 of [7]. In Sections 3.1 and 3.2 above, we have considered necessary conditions of optimality for piecewise continuous controls (see p. 10 and pp. 20-21 of [10]). It remains to show that the measurable optimal control may be taken to be piecewise continuous. This is proven by observing that if we consider the maximum principle for measurable controls†† (see p. 81 of [10]) in the backwards synthesis of extremals, then the optimal control may be taken to

†It has not been possible to determine the optimality of a policy by citing one of the many sets of sufficient conditions that are available (see [2], [14], [20]). In particular, although the planning horizon for the problem at hand is of fixed length, one cannot invoke the sufficient conditions based on convexity of Mangasarian [9] or Funk and Gilbert [3] because the right-hand sides of the differential equations (7) are not concave functions of r_1 and ϕ_1.

††We have taken the liberty of changing the sign of the adjoint vector of Pontryagin et al. [10] (see p. 108 of [2]). When the admissible controls are measurable and bounded, the Hamiltonian (8) only attains its maximum almost everywhere in time.

be piecewise constant (and hence piecewise continuous).†

We will now show that the <u>optimal control must be constant</u>.†† This is done by showing that the returns from both Policy B-B and also Policy S††† for a given point in the initial state space are dominated by the return corresponding to a constant extremal control. We denote the value of the criterion functional corresponding to Policy 0 as J_0, that corresponding to Policy B-B as J_B, etc. Then we have

$$J_0 = \alpha_2 a_2 c_2 \left\{ \left(\frac{r_1^o}{a_1}\right) \frac{R}{c_1} + \left(\frac{r_2^o}{a_2}\right) \frac{1}{c_2} e^{c_2 T} - \left[\frac{RT}{c_1} + \frac{1}{c_2^2} (e^{c_2 T} - 1)\right]\right\}, \tag{31}$$

$$J_1 = \alpha_2 a_2 c_2 \left\{ \left(\frac{r_1^o}{a_1}\right) \frac{R}{c_1} e^{c_1 T} + \left(\frac{r_2^o}{a_2}\right) \frac{1}{c_2} - \left[\frac{R}{c_1^2} (e^{c_1 T} - 1) + \frac{T}{c_2}\right]\right\}, \tag{32}$$

$$\begin{aligned} J_B = \alpha_2 a_2 c_2 \Bigg\{ & \left(\frac{r_1^o}{a_1}\right) \frac{R}{c_1} e^{c_1 (T-\tau_1)} + \left(\frac{r_2^o}{a_2}\right) \frac{1}{c_2} e^{c_2 \tau_1} \\ & - \frac{R}{c_1^2} (e^{c_1(T-\tau_1)} - 1 + c_1 \tau_1) - \frac{1}{c_2^2} ([1 + c_2 (T-\tau_1)] e^{c_2 \tau_1} - 1) \Bigg\}, \end{aligned} \tag{33}$$

and

$$\begin{aligned} J_S = \alpha_2 a_2 c_2 \Bigg\{ & \left(\frac{r_1^o}{a_1}\right) \frac{R^\alpha}{K} e^{KT} - \Bigg[\frac{R}{K^2} (R^{-\beta} e^{KT} - 1) \\ & + \frac{1}{c_1 c_2} R \ln R + \frac{1}{c_2^2} (R-1)\Bigg]\Bigg\}. \end{aligned} \tag{34}$$

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†This follows from the control variable appearing linearly in the Hamiltonian (8), the control variable space being compact, and the switching function being continuous for $0 \le t \le T$. The maximum principle (also singular control considerations) then yields that the optimal control must be peicewise constant almost everywhere, since $S_\phi(t)$ can change sign at most once. Hence, it may be considered to be piecewise constant (see p. 130 of [10]). [The author wishes to thank J. Wingate for pointing out this type of argument.]

††This was first conjectured by Professor Frank Faulkner.

†††By the principle of optimality (see [2]) it suffices for the purpose of showing that a singular solution is always nonoptimal to consider a singular extremal which begins with a singular subarc.

where $\alpha = c_2/(c_1+c_2)$, $\alpha + \beta = 1$, and $K = c_1c_2/(c_1+c_2)$. It is convenient to define $\Delta J_{1-0} = J_1 - J_C$, etc., and then

$$\Delta J_{1-0} = \alpha_2 a_2 c_2 \left\{ R\left[\left(\frac{r_1^o}{a_1}\right)\left(\frac{e^{c_1T}-1}{c_1}\right) - \frac{1}{c_1^2}\left(e^{c_1T}-1-c_1T\right)\right] - \left[\left(\frac{r_2^o}{a_2}\right)\left(\frac{e^{c_2T}-1}{c_2}\right) - \frac{1}{c_2^2}\left(e^{c_2T}-1-c_2T\right)\right]\right\}, \tag{35}$$

$$\Delta J_{1-B} = \alpha_2 a_2 c_2 \left\{ R\left[\left(\frac{r_1^o}{a_1}\right)\left(\frac{e^{c_1T}-e^{c_1(T-\tau_1)}}{c_1}\right) - \frac{1}{c_1^2}\left(e^{c_1T}-e^{c_1(T-\tau_1)}-c_1\tau_1\right)\right] - \left[\left(\frac{r_2^o}{a_2}\right)\left(\frac{e^{c_2\tau_1}-1}{c_2}\right) - \frac{1}{c_2^2}\left(e^{c_2\tau_1}+c_2(T-\tau_1)e^{c_2\tau_1}-1-c_2T\right)\right]\right\}, \tag{36}$$

and[†]

$$\Delta J_{1-S} = \alpha_2 a_2 c_2 \left\{\left(\frac{r_1^o}{a_1}\right)\left[\frac{1}{c_1}\left(Re^{c_1T}-1\right) - \frac{1}{K}\left(R^{\alpha}e^{KT}-1\right)\right] + \frac{R}{K^2}\left(R^{-\beta}e^{KT}-1-\frac{KT}{R}\right) - \frac{R}{c_1^2}\left(e^{c_1T}-1-\frac{c_1T}{R}\right) + \frac{1}{c_1c_2}R\ln R + \frac{1}{c_2^2}(R-1)\right\}. \tag{37}$$

We now state and prove Lemma 1.

> LEMMA 1: Assume that $T \geq \tau_1$. If $\Delta J_{1-0} \geq 0$, then $\Delta J_{1-B} \geq 0$.

PROOF:

(a) We consider for $t \geq \tau_1$

$$F(t) = R\left\{\left(\frac{r_1^o}{a_1}\right)\left(\frac{e^{c_1t}-e^{c_1(t-\tau_1)}}{c_1}\right) - \frac{1}{c_1^2}\left(e^{c_1t}-e^{c_1(t-\tau_1)}-c_1\tau_1\right)\right\} - \left\{\left(\frac{r_2^o}{a_2}\right)\left(\frac{e^{c_2\tau_1}-1}{c_2}\right) - \frac{1}{c_2^2}\left(e^{c_2\tau_1}+c_2(t-\tau_1)e^{c_2\tau_1}-1-c_2t\right)\right\}.$$

[†]In computing ΔJ_{1-S} we assume that $r_1^o/a_1 = r_2^o/a_2$.

Then $\Delta J_{1-0} \geq 0 \Leftrightarrow F(t=\tau_1) \geq 0$.

(b) We compute that

$$F'(t) = R(e^{c_1 t} - e^{c_1(t-\tau_1)})\left\{\left(\frac{r_1^o}{a_1}\right) - \frac{1}{c_1}(1-e^{-c_1\tau_1})\right\} + \frac{1}{c_2}(e^{c_2\tau_1}-1).$$

(c) If $c_1 r_1^o \leq a_1$, then $\frac{dr_1}{dt}(t) \leq 0$ for $0 \leq t \leq t_1$ so that $(r_1^o/a_1) \geq (r_1(t_1)/a_1) \geq \tau_1$. It follows that $F'(t) \geq 0$. If $c_1 r_1^o > a_1$, then $F'(t) > 0$. Thus, we always have $F'(t) \geq 0$ for $t \geq \tau_1$.

(d) By (a) and (c), we have $F(t) \geq 0$, whence follows the lemma. Q.E.D.

> LEMMA 2: For $t_1 = T - \tau_1 \geq 0$, we have $\Delta J_{0-B} \geq 0$ with $\Delta J_{0-B} > 0$ for $t_1 > 0$.

PROOF:

(a) We consider for $t_1 \geq 0$

$$F(t_1) = -R\left\{\left(\frac{r_1^o}{a_1}\right)\left(\frac{e^{c_2t_1}-1}{c_2}\right) - \frac{1}{c_1^2}(e^{c_1t_1}-1-c_1t_1)\right\}$$

$$+ e^{c_2\tau_1}\left\{\left(\frac{r_2^o}{a_2}\right)\left(\frac{e^{c_2t_1}-1}{c_2}\right) - \frac{1}{c_2^2}(e^{c_2t_1}-1-c_2t_1)\right\}.$$

We observe that $F(t_1=0) = 0$.

(b) We compute that $F'(t_1) = -\frac{R}{a_1}\left\{\frac{1}{c_1}[(c_1r_1^o-a_1)e^{c_1t_1}+a_1]\right\} + \frac{e^{c_2\tau_1}}{a_2} \times$ $\{r_2^o e^{c_2t_1} - \frac{a_2}{c_2}(e^{c_2t_1}-1)\}$. Considering (22) and (30), it follows that for $t_1 \geq 0$ we have

$$F'(t_1) = e^{c_2\tau_1}\left\{\left(\frac{r_2^o}{a_2}\right)(c^{c_2t_1}-1) - \frac{1}{c_2}(e^{c_2t_1}-1-c_2t_1)\right\}.$$

(c) Recalling (30) that $r_2^o/a_2 \geq t_1$, we have for $t_1 \geq 0$

$$F'(t_1) \geq e^{c_2\tau_1}\{t_1(e^{c_2t_1}-1) - \frac{1}{c_2}(e^{c_2t_1}-1-c_2t_1)\} \geq 0,$$

since for $t \geq 0$ we have $g(t) \geq 0$, where $g(t) = t(e^{c_2t}-1) - (e^{c_2t}-1-c_2t)/c_2$. This follows from $g(t=0) = 0$ and $g'(t) \geq 0 \;\forall\; t \geq 0$.

(d) Thus, $F(t_1) \geq 0 \;\forall\; t_1 \geq 0$, whence follows the lemma. Q.E.D.

As an immediate consequence of Lemmas 1 and 2 we have Theorem 1.

THEOREM 1: For $T \geq \tau_1 > 0$, we have $\max(J_0,J_1) \geq J_B$ with strict inequality holding for $T > \tau_1$.

LEMMA 3: Assume that $R \geq 1$ and $T \geq \tau_1$. Then we have $\Delta J_{1-S} \geq 0$ with $\Delta J_{1-S} > 0$ for $R > 1$.

PROOF:

(a) We consider for $t \geq 0$

$$F(t) = t\{(Re^{c_1t}-1)/c_1 - (R^\alpha e^{Kt}-1)/K\} + R(R^{-\beta}e^{KT} - 1 - Kt/R)/K^2$$

$$- R(e^{c_1t}-1-c_1T/R)/c_1^2 + (R \ln R)/(c_1c_2) + (R-1)/c_2^2 .$$

Then we have

$$F(t=0) = R(R^{-\beta}-1)/K^2 + (R \ln R)/(c_1c_2) + (R-1)/c_2^2 = f(R) \geq 0$$

with $f(R) > 0$ for $R > 1$. This follows from $f(R=1) = f'(R=1) = 0$ and $f''(R) = (1-R^{-\beta})/(c_1c_2R)$.

(b) Computing $F'(t) = R^{\alpha}t(R^{\beta}e^{c_1t}-e^{Kt}) \geq R^{\alpha}t(e^{c_1t}-e^{Kt}) > 0$ for $R \geq 1$ and $t > 0$, we see from (a) that $F(t;R) \geq 0$ with $F(t;R) > 0$ for $R > 1$.

(c) We now consider $G(t) = (Re^{c_1t}-1)/c_1 - (R^{\alpha}e^{Kt}-1)/K$. It follows that $G(t=0) = 1/c_2 + R/c_1 - R^{\alpha}/K = g(R) \geq 0$, since $g(R=1) = 0$ and $g'(R) = (1-R^{-\beta})/c_1$. Also $G'(t) = R^{\alpha}(R^{\beta}e^{c_1t}-e^{Kt}) \geq 0$. Hence, $G(t) \geq 0$.

(d) Recalling that $r_1^0/a_1 \geq T$, we have by (c) that $\Delta J_{1-S} \geq \alpha_2 a_2 c_2 F(T;R) \geq 0$ with $F(T;R) > 0$ for $R > 1$. Q.E.D.

From Lemma 3 follows Theorem 2.

THEOREM 2: Assume that $R \geq 1$ and $T \geq \tau_1$. Then $\max(J_0,J_1) \geq J_S$ with inequality holding for $R > 1$.

Thus, we see from Theorems 1 and 2 that the optimal control must be constant and equal to either 0 or 1 for $0 \leq t \leq T$. The results shown in Table II and Figures 2 and 3 then follow from consideration of ΔJ_{1-0} (see equation (35)).

4. Development of Optimal Policy for Problem 2.

In this case we consider (1) with the criterion function $J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$. Thus, for this problem the state space (considering time to be an additional state variable) is five dimensional.

4.1. Necessary Conditions of Optimality.

The Hamiltonian [2] is given by (using (2))

$$H = -\sum_{i=1}^{2} p_i a_i y_i - q_1 \phi c_1 y_1 - q_2(1-\phi)c_2 y_2, \tag{38}$$

so that the maximum principle yields the extremal control law

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad S_\phi(t) > 0, \\ 0 & \text{for} \quad S_\phi(t) < 0, \end{cases} \tag{39}$$

where $S_\phi(t)$ denotes the ϕ-switching function defined by

$$S_\phi(t) = c_1(-q_1)y_1 - c_2(-q_2)y_2. \tag{40}$$

The adjoint system of equations (again using (2) for convenience) is given by (assuming that $x_i(T) > 0$)

$$p_i(t) = v_i \quad \text{for} \quad 0 \le t \le T \quad \text{with} \quad i = 1,2,$$

and (41)

$$\dot{q}_i = a_i v_i + \phi_i^* c_i q_i \quad \text{with} \quad q_i(T) = -w_i \quad \text{for} \quad i = 1,2.$$

Computing the first two time derivatives of the switching function

$$\dot{S}_\phi(t) = -a_1 c_1 v_1 y_1 + a_2 c_2 v_2 y_2, \tag{42}$$

$$\ddot{S}_\phi(t) = a_1 c_1 v_1 y_1 (c_1 \phi) - a_2 c_2 v_2 y_2 (c_2(1-\phi)), \tag{43}$$

we see that on a <u>singular subarc</u> we have [2], [6]

$$y_1/y_2 = a_2 c_2 v_2/(a_1 c_1 v_1), \tag{44}$$

$$(-q_1)/(a_1 v_1) = (-q_2)/(a_2 v_2), \tag{45}$$

with the singular control given by

$$\phi_S = c_2/(c_1+c_2). \tag{46}$$

On such a singular subarc the <u>generalized Legendre-Clebsch condition</u> is satisifed, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1 c_1 v_1 y_1 (c_1+c_2) > 0.$

For Problem 1 it was convenient to consider a "reduced" state space of $t, r_1 = x_1/y_1$, and r_2, while for Problem 2 we considered the "full" state space of t, x_1, x_2, y_1, and y_2. It seems appropriate to point out the corresponding relationship between the adjoint variables in these two state spaces. This is easily seen by considering the optimal return function (see [2]), W, and the following transformation of variables

$$t = t, \qquad r_i = x_i/y_i \quad \text{for} \quad i = 1,2. \tag{47}$$

Then we have, for example, $p_i(t) = \dfrac{\partial W}{\partial x_i(t)} = \dfrac{\partial W}{\partial r_i}\dfrac{\partial r_i}{\partial x_i}$ so that we obtain

$$p_i = \lambda_i/y_i \quad \text{and} \quad q_i = -r_i\lambda_i/y_i \quad \text{for} \quad i = 1,2. \tag{48}$$

It seems appropriate to point out that alternatively Problem 1 could have been solved in the "full" state space of t, x_1, x_2, y_1, and y_2, while Problem 2 cannot be solved in the "reduced" state space. The latter follows from consideration of (41) and the requirement (see (48) above) that $p_i/q_i = -1/r_i$ must hold in order for the transformation (47) to be applicable.

4.2. Synthesis of Extremals.

In synthesizing extremals by the usual backwards construction procedure it is convenient to consider

$$S_\phi(\tau=0) = a_2c_2v_2y_2^f\left(\frac{w_1}{a_1v_1}\right)\left\{\frac{a_1c_1v_1y_1^f}{a_2c_2v_2y_2^f} - \left(\frac{w_2}{a_2v_2}\right) \Big/ \left(\frac{w_1}{a_1v_1}\right)\right\}, \tag{49}$$

and

$$\overset{\circ}{S}_\phi(\tau) = a_1c_1v_1y_1 - a_2c_2v_2y_2, \tag{50}$$

where τ denotes the "backwards" time defined by $\tau = T - t$ and $\overset{\circ}{S}_\phi$ denotes the "backwards" time derivative $\overset{\circ}{S}_\phi = dS_\phi/d\tau$. We will omit most of the tedious details of the synthesis of extremals because they are very similar to those given in [14]. Without loss of generality we may assume that $w_1/(a_1v_1) \geq w_2/(a_2v_2)$, and then there are two cases to be considered:

(I) $w_1/(a_1v_1) = w_2/(a_2v_2)$,

(II) $w_1/(a_1v_1) > w_2/(a_2v_2)$.

CASE I: $w_1/(a_1v_1) = w_2/(a_2v_2)$; i.e. $w_i = ka_iv_i$ for $i = 1,2$.

In this case (49) becomes

$$S_\phi(\tau=0) = a_2c_2v_2y_2^f(w_1/(a_1v_1))\{a_1c_1v_1y_1^f/(a_2c_2v_2y_2^f) - 1\},$$

whence follows the synthesis of extremals shown in Figure 4.

CASE II: $w_1/(a_1v_1) > w_2/(a_2v_2)$.

In this case it follows from (39), (49), and (50) that for $\rho^f = y_1^f/y_2^f \geq a_2c_2v_2/(a_1c_1v_1)$ we have $S_\phi(\tau) > 0$ and $\phi^*(\tau) = 1$ for all $\tau > 0$. Since $S_\phi(\tau=0) \leq 0 \Rightarrow \mathring{S}_\phi(\tau=0) < 0$, it follows that for $\rho^f \leq \left(\frac{a_2c_2v_2}{a_1c_1v_1}\right)\left(\frac{w_2}{a_2v_2}\right)/\left(\frac{w_1}{a_1v_1}\right)$ we have $S_\phi(\tau) < 0$ and $\phi^*(\tau) = 0$ for all $\tau > 0$.

There may be a change in sign of $S_\phi(\tau)$ for $c_2w_2/(c_1w_1) < \rho^f < a_2c_2v_2/(a_1c_1v_1)$. In this case $\phi^*(\tau) = 1$ for $0 \leq \tau \leq \tau_1$ and then

$$S_\phi(\tau) = a_2c_2v_2y_2^f\left\{\frac{1}{c_1}(e^{c_1\tau}-1)\left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\rho^f - \tau + \left(\frac{a_1c_1v_1}{a_2c_2v_2}\right)\left(\frac{w_1}{a_1v_1}\right)\rho^f - \left(\frac{w_2}{a_2v_2}\right)\right\}. \quad (51)$$

It is clear that we must have $S_\phi(\tau=\tau_1) \leq 0$. If $S_\phi(\tau=\tau_1) < 0$, then we have a <u>transition surface</u> with τ_1 (denoted as τ_ϕ) given by the smaller of the two positive roots of $G(\tau=\tau_\phi;\rho^f) = 0$, where $G(\tau;\rho^f)$ is given in Table III. If $\mathring{S}_\phi(\tau=\tau_1) = 0$, the <u>singular subarc</u> may be entered, and then we have that τ_1 (denoted at τ_S) is given by the unique nonnegative root of $F(\tau=\tau_S) = 0$, where $F(\tau)$ is given in Table III. We denote the corresponding value of ρ^f as ρ_S^f. Then there is no switch in ϕ^* for $\rho^f > \rho_S^f$. We state this as Theorem 3.

THEOREM 3: $\phi^*(\tau) = 1$ for all $\tau \geq 0$ when $\rho^f > \rho_S^f$.

PROOF: Immediate by $G(\tau=\tau_S;\rho^f=\rho_S^f) = F(\tau=\tau_S) = 0$ and $\partial G/\partial\rho^f > 0$, since then there is no solution to $G(\tau=\tau_1;\rho^f) = 0$ for $\rho^f > \rho_S^f$. Q.E.D.

The bounds on τ_S shown in Table III are developed as follows. First assume that $w_1/(a_1v_1) \leq 1/c_1$. We consider $F(\tau) = \tau + (1/c_1-w_1/(a_1v_1))e^{-c_1\tau} - (1/c_1-w_2/(a_2v_2))$. Then $c_1w_1/(a_1v_1) \leq F'(\tau) \leq 1$ and $F''(\tau) \geq 0$ for $w_1/(a_1v_1) \leq 1/c_1$, whence follow the bounds shown in Table III. Other developments are similar.

The above information immediately leads to the extremal field shown in Figure 5 (see also Tables II and III).

4.3. Determination of the Optimal Fire-Support Policy.

The optimality of the extremal fire-support policy developed above follows according to the reasoning given in Section 3.3 by the uniqueness of extremals.

5. Development of Optimal Policy for Problem 3.

In this case we consider (1) with the criterion functional

$$J_3 = \left(\sum_{k=1}^{2} v_k x_k(T)\right) \Big/ \left(\sum_{k=1}^{2} w_k y_k(T)\right).$$

4.1. Necessary Conditions of Optimality.

The necessary conditions of optimality for Problem 3 are the same as those for Problem 2 except that the boundary conditions for the adjoint variables are different. Thus, (38) through (40) again apply to Problem 3. The adjoint system of equations (again using (2) for convenience) is given by (assuming that $x_i(T) > 0$)

$$p_i(t) = v_i/D \quad \text{for} \quad 0 \leq t \leq T \quad \text{with} \quad i = 1,2,$$

and (52)

$$\dot{q}_i = a_i p_i + \phi_i^* c_i q_i \quad \text{with} \quad q_i(T) = -w_i J_3/D \quad \text{for} \quad i = 1,2,$$

where $D = \sum_{k=1}^{2} w_k y_k(T)$.

Computing the first two time derivatives of the switching function

$$\dot{S}_\phi(t) = -a_1 c_1 p_1 y_1 + a_2 c_2 p_2 y_2, \tag{53}$$

$$\ddot{S}_\phi(t) = a_1 c_1 p_1 y_1 (c_1 \phi) - a_2 c_2 p_2 y_2 (c_2 (1-\phi)), \tag{54}$$

we find that (44) through (46) again hold on a singular subarc. On such a singular subarc the generalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1 c_1 v_1 y_1 (c_1 + c_2)/D > 0$.

5.2. Synthesis of Extremals.

The synthesis of extremals is essentially the same as for Problem 2 (see Section 4.2 above) except that we have

$$S_\phi(\tau=0) = J_3 a_2 c_2 v_2 y_2^f \left(\frac{w_1}{a_1 v_1}\right)\left\{\frac{a_1 c_1 v_1 y_1^f}{a_2 c_2 v_2 y_2^f} - \left(\frac{w_2}{a_2 v_2}\right)/\left(\frac{w_1}{a_1 v_1}\right)\right\}/D, \tag{55}$$

and

$$\overset{\circ}{S}_\phi(\tau) = (a_1 c_1 v_1 y_1 - a_2 c_2 v_2 y_2)/D. \tag{56}$$

It follows that

$$S_\phi(\tau) = \{J_3 a_2 c_2 v_2 y_2^f \left(\frac{w_1}{a_1 v_1}\right)\left[\left(\frac{a_1 c_1 v_1 y_1^f}{a_2 c_2 v_2 y_2^f}\right) - \left(\frac{w_2}{a_2 v_2}\right)/\left(\frac{w_1}{a_1 v_1}\right)\right] + \int_0^\tau (a_1 c_1 v_1 y_1(\sigma) - a_2 c_2 v_2 y_2(\sigma))d\sigma\}/D. \tag{57}$$

5.3 Determination of the Optimal Fire-Support Policy.

As for Problem 2, the optimality of the extremal fire-support policy developed above follows according to the reasoning given in Section 3.3 by the uniqueness of extremals.

6. Future Research Directions.

In this section we discuss possible future research suggested by the work reported in this appendix. First of all there remains computational work to be done on Problems 2 and 3. It should be recalled that in Figures 2 through 5 we qualitatively sketched the optimal fire-support policies for Problems 1 through 3. It was not possible, however, at this time to report actual numerical computations. We would recommend doing this in the future. Of particular interest would be the comparison of switching times in the optimal fire-support policies for Problems 2 and 3 for $w_1/(a_1v_1) > w_2/(a_2v_2)$ to see how model parameter values and force levels affect the timing of changes in fire distribution.

Secondly, it is of interest to study the dependence of the structure of the optimal time-sequential fire-support policy for the W fire-support units (see Figure 1) upon the nature of the criterion function $J = Q(\underset{\sim}{x}_f, \underset{\sim}{y}_f)$, where $\underset{\sim}{x}_f^T$ denotes (x_1^f, x_2^f) with $x_i^f = x_1(T)$, etc. In the work at hand we have examined the consequences for optimal fire-support allocation of several functional forms for the criterion functional. Based on this work it appears worthwhile to examine other functional forms for $Q(\underset{\sim}{x}_f, \underset{\sim}{y}_f)$. It seems reasonable on military grounds to require that

$$\frac{\partial Q}{\partial x_i^f} > 0 \quad \text{and} \quad \frac{\partial Q}{\partial y_i^f} < 0 \quad \text{for all} \quad x_i^f, y_i^f \geq 0.$$

Furthermore, one might postulate either of the following functional forms for $Q(x_f, y_f)$†:

$$\text{(a)} \quad Q(\underset{\sim}{x}_f, \underset{\sim}{y}_f) = F(\underset{\sim}{x}_f) - G(\underset{\sim}{y}_f),$$

or

†Pugh and Mayberry [11] have suggested using the ratio of aggregated force values.

$$\text{(b)} \quad Q(\underset{\sim}{x}_f, \underset{\sim}{y}_f) = F(\underset{\sim}{x}_f)/G(\underset{\sim}{y}_f).$$

It is of interest to study cases in which F (and/or G) is a

(A) concave function,

or (B) convex function,

or (C) quasi-concave function,

or (D) quasi-convex function.

To give a concrete example, as a representative concave function one might consider

$$G(\underset{\sim}{y}_f) = \sum_{i=1}^{2} \beta_i (1 - e^{-\alpha_i y_i^f})/\alpha_i .$$

After the dependence of the structure of the optimal time-sequential fire-support policy upon the functional form of the terminal return $Q(\underset{\sim}{x}_f, \underset{\sim}{y}_f)$ has been studied for the above problem (1), it would seem appropriate to consider a problem like a one-sided version of the "Tactical Air-War Campaign" (see Appendix E of [18]) or the Isbell-Marlow fire-distribution problem (see [12], [20]). Such a research program would lead to a better understanding of the effects on optimizing tactical decisions of the quantification of military objectives and of the valuation of military resources. This in turn would result in a better understanding of the optimization of combat dynamics. In particular, this would hopefully lead to a better understanding of quantitative justification for time-sequential fire-support allocation rules in terms of different quantifications of military objectives.

7. Discussion.

In this section we will discuss what we have learned about the dependence of the structure of optimal time-sequential fire-support policies upon the quantification of military objectives. We studied this dependence by considering three specific problems (each corresponding to a different quantification of objectives (i.e. criterion functional)) for which solutions were developed. We have pointed out above (see Section 2.4) the need for future computational work on these problems. Thus, our remarks here must be limited to a qualitative comparison of the optimal fire-support policies.

The most significant finding is that essentially the entire structure of the optimal time-sequential fire-support policy may be changed by changing the quantification of military objectives. We feel that there are basically two types of military strategies: (1) obtain a "local" advantage and (2) obtain an "overall" advantage. The criterion function for Problem 1 (i.e. $J_1 = \sum_{k=1}^{2} \alpha_k x_k(T)/y_k(T)$, a weighting of force ratios in the two separate combat areas) reflects the striving to attain a "local" advantage (referred to above as a "breakthrough" tactic). The corresponding optimal fire-support policy was to concentrate all supporting fires on one of the enemy units (the quantitative determination of this is given in Table II) for the entire period of fire support.†

On the other hand, the criterion functionals for Problems 2 and 3 (i.e. $J_2 = \sum_{k=1}^{2} v_k x_k(T) - \sum_{k=1}^{2} w_k y_k(T)$, the difference between overall military worths (computed assuming linear utilities) of forces at the time when supporting fires must be lifted, and $J_3 = \left(\sum_{k=1}^{2} v_k x_k(T)\right) / \left(\sum_{k=1}^{2} w_k y_k(T)\right)$, the ratio

†It should be pointed out that perfect information has been assumed for the state variables (i.e. enemy force levels). In the real world where this assumption may not hold, this policy need not be optimal. Other factors that would temper the use of such a policy in the real world are (1) the need to "pin down" enemy forces with supporting fires (i.e. suppressive effects) and (2) the giving of information to the enemy as to exactly where his defenses will be attacked by the concentration of preparatory fires.

of overall military worths) reflect the striving to attain an "overall" advantage (referred to above as an "attrition" tactic which aims to wear down the overall enemy strength). The corresponding optimal fire-support policies for Problems 2 and 3 were qualitatively similar and could involve a <u>splitting of supporting fires</u> between the two enemy troop concentrations. This property of the optimal fire-distribution policy is not present in the solution to Problem 1 and was anticipated in our earlier work on optimal fire distribution against enemy target types which undergo attrition according to a "linear-law" process (see Section 2.1 above) [13], [14]. The criterion functional for this earlier work was the difference between overall military worths of survivors. Thus, we see that the nonconcentration of fires on particular target types is characteristic of optimal time-sequential fire distribution over enemy target types which undergo attrition according to a "linear-law" process <u>with the objective of attaining an "overall" advantage</u>.

We saw that the structures of the optimal time-sequential fire-support policies for Problems 2 and 3 were qualitatively similar, although the timing of changes in the allocation of supporting fires could be appreciably different. Additionally, the functional dependencies of these switching times for the two problems were different. On the other hand, for the particular valuation (computed according to linear utilities) of forces in which each enemy target type was valued in direct proportion to its rate of destruction of value of the opposing friendly forces, the optimal policies were <u>exactly</u> the same for both problems (see Table II). In this case the optimal fire-support policy took a particularly simple form (see Policy A as given by (6)).

When enemy survivors were <u>not valued in direct proportion to their rate of destruction of friendly value</u>, the optimal policy was more complex (see

Tables II and III or Figure 5). In this case for purposes of describing the optimal fire-support allocation rule, the planning horizon could be considered to be divided into two phases. Moreover, the lengths of these two phases depended on different factors for these two problems. When the planning objective was the maximization of the difference in total military worth of the two forces at the end of the "approach to contact," the length of, for example, PHASE II depended only on the attrition-rate coefficients and enemy force levels and was independent of the attacking-force force levels. However, when the ratio of the total military worths of the two forces was considered (i.e. for Problem 3), the length of PHASE II also depended directly on the attacking friendly force levels.

Thus, we see that (at least for the relatively simple fire-support allocation problem considered here) the nature of the optimal time-sequential allocation policy is strongly influenced by the quantification of military objectives. We hope that as a result of our investigation reported here a better understanding of optimal time-sequential fire-support strategies (in particular, how they depend on combatant objectives) has been developed. We conclude that more work needs to be done on the identification of criteria for making tactical decisions and on the quantification of such criteria. We would propose to ONR for future research the further study of the influence of different quantifications of military objectives on optimal time-sequential fire-support strategies.

REFERENCES

[1] H. Brackney, "The Dynamics of Military Combat," Opns. Res. 7, 30-44 (1959).

[2] A. Bryson and Y. C. Ho, Applied Optimal Control, Blaisdell Publishing Company, Waltham, Massachusetts, 1969.

[3] J. Funk and E. Gilbert, "Some Sufficient Conditions for Optimality in Control Problems with State Space Constraints," SIAM J. Control 8, 498-504 (1970).

[4] A. Karr, "Stochastic Attrition Models of Lanchester Type," Paper P-1030, Institute for Defense Analysis, June 1974.

[5] Y. Kawara, "An Allocation Problem of Fire Support in Combat as a Differential Game," Opns. Res. 21, 942-951 (1973).

[6] H. Kelley, R. Kopp and H. Moyer, "Singular Extremals," pp. 63-101 in Topics in Optimization, G. Leitman (Ed.), Academic Press, New York, 1967.

[7] E. Lee and L. Markus, Foundations of Optimal Control Theory, John Wiley & Sons, Inc., New York, 1967.

[8] O. Mangasarian, "Sufficient Conditions for the Optimal Control of Nonlinear Systems," SIAM J. Control 4, 139-152 (1966).

[9] S. Moglewer and C. Payne, "A Game Theory Approach to Logistics Allocation," Naval Res. Log. Quart. 17, 87-97 (1970).

[10] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze and E. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York, 1962.

[11] G. Pugh and J. Mayberry, "Theory of Measures of Effectiveness for General-Purpose Military Forces: Part I. A Zero-Sum Payoff Appropriate for Evaluating Combat Strategies," Opns. Res. 21, 867-885 (1973).

[12] J. Taylor, "On the Isbell and Marlow Fire Programming Problem," Naval Res. Log. Quart. 19, 539-556 (1972).

[13] J. Taylor, "Lanchester-Type Models of Warfare and Optimal Control," Naval Res. Log. Quart. 21, 79-106 (1974).

[14] J. Taylor, "Target Selection in Lanchester Combat: Linear-Law Attrition Process," Naval Res. Log. Quart. 20, 673-697 (1973).

[15] J. Taylor, "Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent Attrition-Rate Coefficients," Naval Res. Log Quart 21, 683-704 (1974).

[16] J. Taylor, "Survey on the Optimal Control of Lanchester-Type Attrition Processes," to appear in Proceedings of the Conference on the State-of-the-Art of Mathematics in Combat Models (also Naval Postgraduate School Technical Report No. NPS55Tw74031, March 1974).

[17] J. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part II," Naval Postgraduate School Technical Report No. NPS55Tw7211A, November 1972.

[18] J. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part III," Naval Postgraduate School Technical Report No. NPS55Tw74051, May 1974.

[19] J. Taylor, "Appendices C and D of 'Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part III'," Naval Postgraduate School Technical Report No. NPS55Tw74112, November 1974.

[20] J. Taylor, "On the Treatment of Force-Level Constraints in Time-Sequential Combat Problems," Naval Res. Log. Quart. 22, 617-650 (1975).

[21] H. Weiss, "Lanchester-Type Models of Warfare," pp. 82-98 in Proc. First International Conf. Operational Research, John Wiley & Sons, Inc., New York, 1957.

[22] H. Weiss, "Some Differential Games of Tactical Interest and the Value of a Supporting Weapon System," Opns. Res. 7, 180-196 (1959).

APPENDIX C: Optimization of Time-Phased Combat

1. Introduction.

In this appendix we will briefly consider methodology for the development of optimal time-sequential fire-support policies when there are different combat dynamics (i.e. Lanchester-type equations) in different "phases" of a battle. We will consider cases in which the system equations (i.e. the right-hand sides of the differential equation combat model) are discontinuous at an interior point of the planning horizon determined by a given condition on the state variables being satisfied (see pp. 104-105 of [1]). We have chosen to call this situation "time-phased" combat. Such a situation occurs when a breakpoint (determined by a given condition between the state variables being satisfied) is reached.

We will briefly outline how to determine the optimal time-sequential policy (see pp. 104-105 of [1]) for such a model. We will present some preliminary results here and would propose to ONR the further study of these problems as a future research task. We will consider the optimal allocation of supporting fires for an assault by friendly infantry forces within the context of the scenario previously described in Appendix A. Two situations that we will consider here are

(a) breakpoint for the defenders,

(b) breakpoint for the attackers.

2. Breakpoint for Defenders.

Let us consider the attack by heterogeneous X ground forces (infantry) upon the static defensive position of heterogeneous Y ground forces along a "front." The basic scenario of this situation has been described in detail in Section 6.2 of Appendix A and need not be repeated here. The combat situation is shown diagrammatically in Figure 1.

If we consider "breakpoints" for the defending ground combat units,† then our basic combat optimization problem becomes

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(t_f) - \sum_{k=1}^{2} w_k y_k(t_f) \right\},$$

with stopping rule: $t_f - T = 0$,

$$\begin{aligned} &\text{subject to: (battle dynamics)} \quad \frac{dx_i}{dt} = \begin{cases} -a_i y_i & \text{for} \quad y_i > y^i_{BP}, \\ 0 & \text{for} \quad y_i \le y^i_{BP}, \end{cases} \\ &\frac{dy_i}{dt} = -\phi_i c_i y_i \quad \text{for} \quad i = 1,2, \end{aligned} \tag{1}$$

$x_i, y_i \geq 0$ (State Variable Inequality Constraints)

$\phi_1 + \phi_2 = 1$ and $\phi_i \geq 0$ for $i = 1,2$ (Control Variable Inequality Constraints),

where all symbols are as defined in Appendix A (see Section 4 of Appendix A). It will be convenient to consider the single control variable ϕ defined by

$$\phi = \phi_1 \quad \text{so that} \quad \phi_2 = (1-\phi) \quad \text{and} \quad 0 \le \phi \le 1. \tag{2}$$

For $T < +\infty$ it follows that we will always have $y_i(t) > 0$ for $i = 1,2$. Thus, the only state variable inequality constraints (SVIC's) that must be considered are $x_i \geq 0$. However, let us further <u>assume that the attacker's infantry force levels are never reduced to zero</u>. This might be militarily justified on the grounds that X would never attack the Y_i position if his attacking X_i forces could not survive the "approach to contact." Moreover, we will relax this assumption in the next section.

†In other words, a defending unit becomes ineffective upon reaching a given force level.

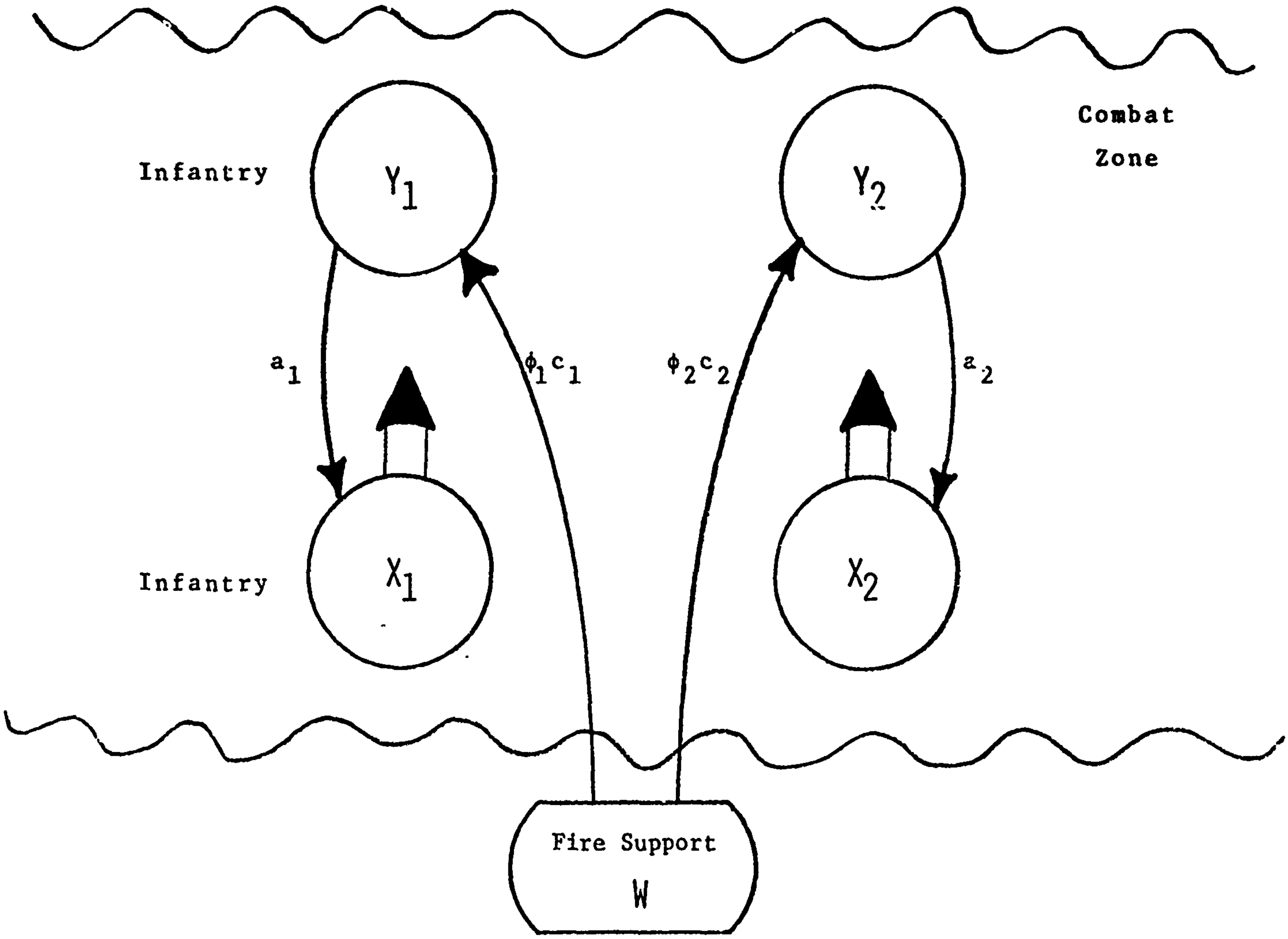

Figure 1. Diagram of Time-Sequential Fire-Support Problem for an Attack of Friendly Forces.

Let us now briefly outline the necessary conditions of optimality (see [1], [3], or [5]) for the above optimal control problem (1). We will see that the basic structure of the optimal time-sequential fire-support policy developed in Section 6.2 of Appendix A is modified "near a breakpoint." We hope to give a more complete treatment of such problems in the future.

For convenience, let us focus on the case in which Y_1 reaches its "breakpoint." This happens at t_1 when

$$\psi^{(1)} = y_1(t_1) - y_{BP}^1 = 0. \tag{3}$$

In the development of necessary conditions of optimality it is convenient to define two phases of combat: PHASE I during which $y_i > y_{BP}^i$ for $i = 1,2$, and PHASE II during which $y_1 \leq y_{BP}^1$ and $y_2 > y_{BP}^2$. We denote the Hamiltonian of PHASE I as $H^{(1)}$ and similarly for $H^{(2)}$.

During <u>PHASE I</u> the Hamiltonian is given by [1] (using (2))

$$H^{(1)} = -\sum_{i=1}^{2} p_i a_i y_i - q_1 \phi c_1 y_1 - q_2 (1-\phi) c_2 y_2, \tag{4}$$

so that the maximum principle yields the extremal control law

$$\phi^*(t) = \begin{cases} 1 & \text{for} \quad S_\phi(t) > 0, \\ 0 & \text{for} \quad S_\phi(t) < 0, \end{cases} \tag{5}$$

where $S_\phi(t)$ denotes the ϕ-switching function defined by

$$S_\phi(t) = c_1(-q_1)y_1 - c_2(-q_2)y_2. \tag{6}$$

The adjoint system of equations for the dual variables (again using (2) for convenience) is given by (assuming that $x_i > 0$)

$$p_i(t) = v_i \qquad \text{for} \quad 0 \leq t \leq T,$$

and

$$q_i = a_i v_i + \phi_i^* c_i q_i \quad \text{with} \quad q_i(T) = -w_i \quad \text{for} \quad i = 1,2. \tag{7}$$

Computing the first two time derivatives of the switching function (6)

$$\dot{S}_\phi(t) = -a_1c_1v_1y_1 + a_2c_2v_2y_2, \tag{8}$$

$$\ddot{S}_\phi(t) = a_1c_1v_1y_1(c_1\phi) - a_2c_2v_2y_2(c_2(1-\phi)), \tag{9}$$

we see that on a <u>singular subarc</u> we have [1]

$$y_1/y_2 = a_2c_2v_2/(a_1c_1v_1), \tag{10}$$

$$(-q_1)/(a_1v_1) = (-q_2)/(a_2v_2), \tag{11}$$

with the singular control given by

$$\phi_S = c_2/(c_1+c_2). \tag{12}$$

On such a singular subarc the generalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1c_1v_1y_1(c_1+c_2) > 0.$

During <u>PHASE II</u> when $y_1 \leq y_{BP}^1$ and $y_2 > y_{BP}^2$ the Hamiltonian is given by

$$H^{(2)} = -p_2a_2y_2 - q_1\phi c_1y_1 - q_2(1-\phi)c_2y_2. \tag{13}$$

The maximum principle again yields the extremal control law (5). The adjoint system is given by (assuming that $x_i > 0$)

$$\begin{aligned} p_i(t) &= v_i && \text{for} \quad 0 \leq t \leq T, \\ \dot{q}_1 &= \phi^* c_1 q_1 && \text{with} \quad q_1(T) = -w_1, \end{aligned} \tag{14}$$

and

$$\dot{q}_2 = a_2v_2 + (1-\phi^*)c_2q_2 \quad \text{with} \quad q_2(T) = -w_2.$$

Computing the first time derivative of the switching function (6)

$$\dot{S}_\phi(t) = -a_2c_2v_2y_2 < 0,$$

we see that singular subarcs are impossible during PHASE II.

At a juncture (defined by (3)) between the two "phases" of battle, we have (see pp. 104-105 of [1])

$$(-q_1(t_1^-)) = (-q_1(t_1^+)) + \xi, \tag{15}$$

$$q_2(t_1^-) = q_2(t_1^+), \tag{16}$$

and

$$p_i(t_1^-) = p_i(t_1^+) \quad \text{for} \quad i = 1,2. \tag{17}$$

The condition that $H^{(1)}(t_1^-) = H^{(2)}(t_1^+)$ (recalling our assumption that $x_1(T) > 0$) yields

$$\phi^*(t_1^-)S_\phi(t_1^-) = \phi^*(t_1^+)S_\phi(t_1^+) + a_1v_1y_1 > 0, \tag{18}$$

so that

$$\phi^*(t_1^-) = 1 \quad \text{and} \quad S_\phi(t_1^-) > 0. \tag{19}$$

We also obtain from (6), (15), and (16) that

$$S_\phi(t_1^-) = S_\phi(t_1^+) + c_1y_1\xi. \tag{20}$$

By (18), (19), and (20) we have

$$y_1(c_1\xi - a_1v_1) = -S_\phi(t_1^+)\{1-\phi^*(t_1^+)\} \geq 0,$$

so that

$$\xi \geq a_1v_1/c_1 > 0. \tag{21}$$

From (15) and (21) we see that the value of members of the Y_1 force is decreased when the unit becomes "ineffective."

Thus, we have proven

THEOREM 1: In the case in which the Y_1 forces reach their "breakpoint" and become ineffective, the optimal time-sequential policy for W is to concentrate all supporting fires on Y_1 (at least for some time immediately preceding the reaching of the "breakpoint").

In other words, the singular subarc is nonoptimal for reaching the Y_1 "breakpoint." Militarily this means that all supporting fires are concentrated on Y_1 in order to make the unit "break" even when the W fire-support units cause attrition to Y_1 according to a "linear-law" process. Thus, we have another (see [9]) quantitative justification of one of the most significant and oft-quoted of Napoleon Bonaparte's sayings (see p. 117 of [4])-- "The principles of war are the same as those of a siege; fire must be concentrated at one point."

3. Breakpoint for Attackers.

We will now consider the case in which one of the attacking units "breaks." Let us consider the same scenario as considered in the previous section. We assume that when the attacking X_i force reaches its "breakpoint," it abandons the attack and withdraws at a rate r_i until it totally disengages the Y_i force.[†] Then our combat optimization problem becomes

$$\underset{\phi_i(t)}{\text{maximize}} \left\{ \sum_{k=1}^{2} v_k x_k(t_f) - \sum_{k=1}^{2} w_k y_k(t_f) \right\},$$

with stopping rule:[††] $t_f - T = 0$,

subject to:
$$\frac{dx_i}{dt} = \begin{cases} -a_i y_i & \text{for} \quad x_i > x_{BP}^i, \\ -\alpha a_i y_i - r_i & \text{for} \quad 0 < x_i \leq x_{BP}^i, \\ 0 & \text{for} \quad x_i = 0, \end{cases}$$

$$\frac{dy_i}{dt} = -\phi_i c_i y_i \qquad \text{for} \quad i = 1,2,$$

$$x_i, y_i \geq 0, \quad \phi_1 + \phi_2 = 1, \quad \text{and} \quad \phi_i \geq 0 \quad \text{for} \quad i = 1,2.$$

[†]During this disengagement time, the fire effectiveness of the defending Y_i force is modified by the factor α.

[††]We assume that X will not launch the attack if both his units will be repulsed by the Y forces.

We observe that the system dynamics are such that the SVIC's are always satisfied. As above, it will sometimes be convenient to consider the single control variable ϕ defined by (2).

Let us now briefly outline the development of the necessary conditions of optimality for the above optimal control problem (22). We will see that the optimal time-sequential fire-support policy depends on the "outcome" of battle and that "local optima" are yielded by the necessary conditions. We hope to give a more complete treatment of such problems in the future. For now we will partially synthesize extremals in one special case.

For convenience, let us focus on the case in which X_1 reaches its "breakpoint." This happens at t_1 when

$$\psi^{(1)} = x_1(t_1) - x_{BP}^1 = 0. \tag{23}$$

The disengagement of the X_1 forces becomes complete at t_2 defined by

$$\psi^{(2)} = x_1(t_2) = 0. \tag{24}$$

Consequently, in the development of necessary conditions of optimality it is convenient to define three phases of combat: PHASE I during which $x_i > x_{BP}^i$ for $i = 1,2$, PHASE II during which $0 < x_1 \leq x_{BP}^1$ and $x_2 > x_{BP}^2$, and PHASE III during which $x_1 = 0$ and $x_2 > x_{BP}^2$. We denote the Hamiltonian of PHASE I as $H^{(1)}$ and similarly for $H^{(2)}$ and $H^{(3)}$.

During PHASE I the Hamiltonian is given by [1] (using (2))

$$H^{(1)} = -\sum_{i=1}^{2} p_i a_i y_i - q_1 \phi c_1 y_1 - q_2 (1-\phi) c_2 y_2. \tag{25}$$

The maximum principle again yields (5) as the extremal control law, and the adjoint equations are

$$p_i(t) = \text{constant},$$

and (26)

$$\dot{q}_i = a_i p_i + \phi_i^* c_i q_i \quad \text{with} \quad q_i(T) = -w_i \quad \text{for} \quad i = 1,2.$$

Computing the first two time derivatives of the switching function (6)

$$\dot{S}_\phi(t) = -a_1 c_1 p_1 y_1 + a_2 c_2 p_2 y_2, \tag{27}$$

$$\ddot{S}_\phi(t) = a_1 c_1 p_1 y_1 (c_1 \phi) - a_2 c_2 p_2 y_2 (c_2 (1-\phi)), \tag{28}$$

we see that on a <u>singular subarc</u> we have

$$y_1/y_2 = a_2 c_2 p_2 / (a_1 c_1 p_1), \tag{29}$$

$$(-q_1)/(a_1 p_1) = (-q_2)/(a_2 p_2), \tag{30}$$

with the singular control given by

$$\phi_S = c_2/(c_1+c_2). \tag{31}$$

On such a singular subarc the generalized Legendre-Clebsch condition is satisfied, since $\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = a_1 c_1 p_1 y_1 (c_1+c_2) > 0.$

During <u>PHASE II</u> when $0 < x_1 \le x_{BP}^1$ and $x_2 > x_{BP}^2$ the Hamiltonian is given by

$$H^{(2)} = -p_1(\alpha a_1 y_1 + r_1) - p_2 a_2 y_2 - q_1 \phi c_1 y_1 - q_2 (1-\phi) c_2 y_2. \tag{32}$$

The maximum principle again yields the extremal control law (5). The adjoint system is given by

$$p_i(t) = \text{constant},$$

$$\dot{q}_1 = \alpha a_1 p_1 + \phi^* c_1 q_1 \qquad \text{with} \quad q_1(T) = -w_1, \tag{33}$$

and

$$\dot{q}_2 = a_2 p_2 + (1-\phi^*) c_2 q_2 \quad \text{with} \quad q_2(T) = -w_2.$$

Computing the first two time derivatives of the switching function (6)

$$\dot{S}_\phi(t) = -\alpha a_1c_1p_1y_1 + a_2c_2p_2y_2, \tag{34}$$

$$\ddot{S}_\phi(t) = \alpha a_1c_1p_1y_1(c_1\phi) - a_2c_2p_2y_2(c_2(1-\phi)), \tag{35}$$

we see that on a <u>singular subarc</u> we have

$$y_1/y_2 = a_2c_2p_2/(\alpha a_1c_1p_1), \tag{36}$$

$$(pq_1)/(\alpha a_1p_1) = (-q_2)/(a_2p_2), \tag{37}$$

with the singular control again given by (31). The generalized Legendre-Clebsch condition is readily seen to hold.

At juncture time t_1 (defined by (23)) between PHASE I and PHASE II of battle, we have (see pp. 104-105 of [1])

$$p_1(t_1^-) = p_1(t_1^+) + \xi, \tag{38}$$

$$p_2(t_1^-) = p_2(t_1^+), \tag{39}$$

and

$$q_i(t_1^-) = q_i(t_1^+) \quad \text{for} \quad i = 1,2. \tag{40}$$

The condition that $H^{(1)}(t_1^-) = H^{(2)}(t_1^+)$ yields

$$\xi = p_1(t_1^+)\{r_1/(a_1y_1)-(1-\alpha)\}. \tag{41}$$

During PHASE III when $x_1 = 0$ and $x_2 > x_{BP}^2$ the Hamiltonian is given by

$$H^{(3)} = -p_2a_2y_2 - q_1\phi c_1y_1 - q_2(1-\phi)c_2y_2, \tag{42}$$

with the maximum principle again yielding the extremal control law (5). The adjoint equations are

$$p_i(t) = \text{constant},$$

$$\dot{q}_1 = \phi^* c_1q_1 \qquad \text{with} \quad q_1(T) = -w_1 \tag{43}$$

and

$$\dot{q}_2 = a_2 p_2 + (1-\phi^*) c_2 q_2 \quad \text{with} \quad q_2(T) = -w_2.$$

Singular subarcs are impossible, since $\dot{S}_\phi(t) = -a_2 c_2 p_2 y_2 < 0$.

At juncture time t_2 (defined by (24)) between PHASE II and PHASE III of battle, we have

$$p_1(t_2^-) = \xi, \quad p_2(t_2^-) = p_2(t_2^+), \quad \text{and} \quad q_i(t_2^-) = q_i(t_2^+) \quad \text{for } i = 1,2. \tag{44}$$

The condition that $H^{(2)}(t_2^-) = H^{(3)}(t_2^+)$ yields

$$p_1(t_2^-) = \xi = 0. \tag{45}$$

In synthesizing extremals by the usual backwards sweep method (see [7] or [8]), there are three cases to be considered (we always assume that $x_2^f = x_2(T) > x_{BP}^2$ so that $p_2(t) = v_2$):

(1) $x_1^f > x_{BP}^1$,

(2) $0 < x_1^f \le x_{BP}^1$,

(3) $x_1^f = 0$.

<u>For Case 1: $x_1^f > x_{BP}^1$</u>, the optimal fire-support policy is the same as that for Problem 2 of Appendix A.

<u>For Case 2: $0 < x_1^f \le x_{BP}^2$</u>, we have $p_1(t) = v_1$. The battle consists of both PHASE I and also PHASE II. Singular subarcs are possible during both phases. The relative position of "singular surfaces" in the state space for the two phases of battle depends on the parameter α (e.g whether or not $\alpha > 1$). Details may be worked out by the usual backwards sweep method.

<u>For Case 3: $x_1^f = 0$</u>, we have $p_1(t) = 0$ for $0 \le t \le T$. The battle consists of PHASE I, PHASE II, and PHASE III. There are no singular subarcs

in the solution when $x_1^f = 0$. If $w_i = ka_i v_i$ (i.e. enemy survivors valued in direct proportion to their rate of destroying friendly value), then

$$S_\phi(\tau=0) = ka_1c_1v_1y_2^f\{y_1^f/y_2^f-a_2c_2v_2/(a_1c_1v_1)\}. \tag{46}$$

The usual arguments now yield that

$$\phi^*(t) = 0 \quad \text{for} \quad 0 \leq t \leq T \quad \text{when} \quad y_1^f/y_2^f \leq \rho_D^f,$$

where $\rho_D^f = a_2c_2v_2/(a_1c_1v_1)$. For $y_1^f/y_2^f > \rho_D^f$, we have

$$\phi^*(\tau) = 1 \quad \text{for} \quad 0 \leq T < \tau_1,$$

where $\tau = T - t$ denotes the "time to go" in the battle and $\tau_1 = ka_1c_1v_1/(a_2c_2v_2)\{y_1^f/y_2^f-a_2c_2v_2/(a_1c_1v_1)\}$. It may be shown that $T \geq t_2 \Rightarrow x_1^f = 0$, where t_2 is given by

$$\alpha x_1^o + (1-\alpha)x_{BP}^1 - r_1(t_2-t_1) - (1-e^{-c_1t_2})\alpha a_1y_1^o/c_1 = 0,$$

and

$$t_1 = \frac{1}{c_1} \ln\left\{\frac{a_1y_1^o}{a_1y_1^o-c_1(x_1^o-x_{BP}^1)}\right\}.$$

Thus, we see that the structure of the optimal time-sequential fire-support policy depends on the "outcome" of battle (e.g. the value of x_1^f). The dependence of the structure of the optimal policy on initial force levels is complicated and remains to be determined in the future. We have seen that one must consider "breakpoints" and different combat dynamics in different phases of battle to insure "realistic" combat situations (such as force levels remaining nonnegative). The determination of the optimal time-sequential fire-support policy for such problems is much more complicated than that for problems previously considered by us. We hope to give this important subject further consideration in the future.

4. Summary.

In this appendix we hav briefly considered the determination of optimal time-sequential fire-support strategies for battles with different combat dynamics in different "phases" of battle. An important instance in which one must consider such a model is when "breakpoints" (see [2] or [6]) are considered for units. In such cases we have seen that the determination of the optimal policy is much more complicated than that for the problems that we have previously considered (see, for example, [7], [8], [9]). Moreover, the structure of the optimal policy was different for the problems considered here than that for the version previously considered in Appendix A.

Thus, the models considered here do lead to a significantly different structure for optimal time-sequential allocation policies than those we have previous considered. "Global" considerations (i.e. which end states of battle can be reached by extremals from a given point in the initial state space) appear to be especially important in developing solutions to such problems. We have only briefly considered these problems here and hope to give them more detailed treatment in the future.

REFERENCES

[1] A. Bryson and Y. C. Ho, Applied Optimal Control, Blaisdell Publishing Co., Waltham, Massachusetts, 1969.

[2] R. Helmbold, "Decision in Battle: Breakpoint Hypotheses and Engagement Termination Data," R-772-PR, The RAND Corporation, June 1971.

[3] E. Lee and L. Markus, Foundations of Optimal Control Theory, John Wiley & Sons, Inc., New York, 1967.

[4] B. H. Liddell Hart, Strategy, Second Revised Edition, Frederick A. Praeger, Inc., New York, 1967.

[5] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze and E. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York, 1962.

[6] S. Spring and S. Miller, "FAST-VAL: Relationships Among Casualties, Suppression, and the Performance of Company-Sized Units," RM-6268-PR, The RAND Corporation, March 1970.

[7] J. Taylor, "Target Selection in Lanchester Combat: Linear-Law Attrition Process," Naval Res. Log. Quart. 20, 673-697 (1973).

[8] J. Taylor, "Lanchester-Type Models of Warfare and Optimal Control," Naval Res. Log. Quart. 21, 79-106 (1974).

[9] J. Taylor, "Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent Attrition-Rate Coefficients," Naval Res. Log. Quart. 21, 683-704 (1974).